The role of domain restrictions in mechanism design: ex post incentive compatibility and Pareto efficiency

by

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Abstract: The possibility of designing efficient, ex-post incentive compatible, single valued, direct mechanisms depends crucially on the domain of types and preferences on which they are defined. In a framework that allows for interdependent types, we identify two classes of domains. For those called knit, we show that only constant mechanisms can be ex post (or even interim) incentive compatible. Then we prove that ex post incentive compatible mechanisms defined on domains called partially knit are efficient. For private values, this implies that strategy proof mechanisms will also be group strategy-proof. We provide voting, matching, auction examples where our theorems apply.

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1 Introduction

A major concern when designing economic mechanisms is to provide agents with incentives to reveal their true characteristics. Setting aside some obviously unsatisfactory solutions, it is well understood that attaining this objective is not always possible. Moreover, when it is, a conflict often arises between the mechanisms' efficiency and incentive compatibility. These generic statements hold for different formulations of the mechanism design problem, and for various concepts of equilibrium. In this paper we concentrate in the possibility of designing ex-post incentive compatible and Pareto efficient direct mechanisms. As a result, we do not need to introduce Bayesian updating, and can work in a framework where agents' preferences are ordinal.[1]

We start from the remark that a mechanism can only meet interesting lists of desiderata if its domain of definition is somehow restricted, and we identify domains on which satisfactory direct mechanisms can be defined, and others where they cannot.[2] In many cases, restricted domains arise from assumptions regarding the structure of alternatives, the dimensionality of types, or the properties of utility functions. And these assumptions, in turn, are often suggested by the applications that the modeler has in mind. Our restricted domains are defined quite differently. We require that if one domain contains certain type profiles, then some other well-defined profiles must also belong to it. The classes of domains that we'll now describe informally, and rigorously define in the next section, are suggested from a careful analysis of a variety of possibility and impossibility results that arise in different fields of application and may look quite unrelated at first glance. Our approach allows us to abstract from the specifics and to identify essential and common characteristics of domains, which define much of the frontier between possibility and impossibility.

We consider the general case where agents' types, hence their preferences, can be interdependent. In that case, the profile of individual preferences may only be fully determined once all agents know the joint profile of types, through what we call the preference function.

Let's be specific about the demands we impose on mechanisms, in order to consider them satisfactory. One first attractive and well-studied requirement is that of ex post incentive compatibility, guaranteeing truthful revelation of types to be a Nash equilibrium in all the games that result from any specification of possible type profiles. We also introduce a second concept of ex post group incentive compatibility under which truthful revelation is required to be a strong Nash equilibrium. And we impose an additional condition on mechanisms that we call respectfulness, which will be needed for some of our applications but will trivially hold in other cases.

[1] The study of incentive compatibility in Bayesian terms was started by d'Aspremont and Gérard-Varet (1979), and Arrow (1979), and its appropriate formulation and results depend on the information that will be available to the agents at the time where the analysis is carried out. The case of interdependent types was first studied by D'Aspremont, Crémer, and Gérard-Varet (1990). The notion of ex post incentive compatibility corresponds to the time where agents have received all possible information, and can be defined without attributing cardinal utility to agents, as it does not require Bayesian update. See Jackson (2003).

[2] The expression "domain restriction" is mostly used in social choice theory, where type profiles are in fact preference profiles. But the need to limit attention to certain subsets of types or preferences also holds for other environments, where the word domain restriction is less standard. For example, assumptions of convexity, separability, continuity or others can act as restrictions on other kinds of mechanisms.

Here is an informal overview of our definitions and results.

We say that a domain of type profiles is knit if it is possible to connect all pairs of profiles it contains through specific sequences of type transformations, whose induced changes in preference profiles meet adequate conditions. A domain is partially knit if it allows the same sort of connection between a smaller but still well defined subset of pairs of its type profiles.[3]

We present two main results, each one based on the consideration of domains satisfying one of our two conditions.

Our Theorem 1 is in the vein of impossibility results. It states that only the constant mechanism can be ex post incentive compatible and respectful[4] if its domain of definition is knit. In fact, the result only applies to the case of interdependent preferences because, as we prove later on, no domain of types can be knit in the particular case of private values. The informed reader will observe that the conclusion of our theorem is the same that was obtained by Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006), but the analogy stops here, since the context and the assumptions in each case are very different. Also notice that, since we work with single valued direct mechanisms, our environments are separable in the sense of Bergemann and Morris (2005), and their Corollary 1 applies: no mechanism is interim incentive compatible unless it is ex post incentive compatible. Because of that, our Theorem 1 has direct implications on the weaker interim notion, with no need to be explicit about agents' beliefs.

Our Theorem 2 can be read as being positive or negative, depending on the specific context of application. It states that in partially knit domains all respectful and ex post incentive compatible mechanisms will also be ex post incentive compatible. Notice that, unlike our previous result, this one applies both in the case of interdependent types, and also in that of private values. Since ex post group incentive compatibility implies Pareto efficiency on the range, the theorem points at the fact that the generally assumed incompatibility between efficiency and the provision of good incentives may be sometimes avoided, under circumstances whose interest depends crucially on the domains of types and on the range of the functions under consideration. For example, in domains for which the only admissible ex post incentive compatible mechanisms are dictatorial, hence Pareto efficient, the conclusion of our theorem, although true, is not interesting. But there are also interesting partially knit domains admitting attractive ex post incentive compatible mechanisms. For those, Theorem 2 explains why Pareto efficiency can be achieved, along with good incentive properties. It is also important to recall that, in private values cases, ex post incentive compatibility is equivalent to strategy-proofness. Likewise, ex post group incentive compatibility becomes equivalent to strong group strategy-proofness. Hence, a corollary for the case of private values is that, under the conditions of our second theorem, individual and strong group strategy-proofness become equivalent. This parallels results that we obtained in Barberà,

[3]The purpose of our introduction is to present the reader with a general roadmap. The details regarding what we exactly mean by the terms connecting pairs of type profiles, or adequate conditions are provided in the formal definitions in Section 2, and clearly illustrated in the analysis of examples of applicatons in Appendix B.

[4]Again, we leave the definition of respectfulness for Section 2, and provide examples of mechanisms satisfying it in Section 4 and Appendix B.

Berga, and Moreno (2010, 2016) connecting individual and weak group strategy-proofness.[5]

Our discussion has been abstract till now, but we already said that our results are based in a careful analysis of a variety of problems that arise in different settings, and in specific models that are inspired by essential contributions to several fields of application. We illustrate this by providing examples of situations where our results apply. The examples come in pairs. Two of them refer to deliberative juries and are inspired in our reading of Austen-Smith and Feddersen (2006). Another two address the problem of assigning indivisible objects as in Che, Kim, and Kojima (2015). The last two examples refer to auctions, following the trail of Dasgupta and Maskin (2000) and Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006).

We attach much importance to these examples for several reasons.

One reason is that they show the unifying power of our approach. The models we get inspiration from look very different from each other, because they describe the types of agents in terms that are specific to the specific application. Yet our conditions and conclusions apply to all of them at a time. This is because we have arrived at the abstract formulation of our domain restrictions by scrutinizing what is common in the nature of these settings, and many others, for which results about ex-post incentive compatibility and related concepts had been carefully explored.

A second reason is that, in each of the applications, we can provide blood and flesh to the general and rather abstract notion of a preference function. We do that by identifying as a special part of each agent's type the specific rules that translate the information contained in the profile of types into the preferences of that agent. This separation is unnecessary for the validity of our general results, but it allows us to be specific about the different considerations that lead to the relevant preference functions considered in each of our examples.

A third and very important reason to present the examples in pairs is because they allow us to show that the frontier between worlds where impossibility prevails, and others where ex-post incentive compatibility is compatible with a good degree of efficiency can be surprisingly thin. For each one of our fields of application, we present examples that look rather similar and yet belong to one of these worlds or to the other, depending on whether their domain of definition is or is not knit. Since knit domains are also partially knit, our Theorem 2 applies also there, if only to the constant function. But our examples clarify that attractive mechanisms may exist on partially knit ranges.

We do not claim that checking for our domain conditions will always be easy, though it may be quite feasible in some cases. At any rate, that is not the purpose of our exercise. Rather, we want to prove that the essence of difficulties or the possibilities that one encounters when designing ex post incentive compatible mechanisms lies in the nature of restrictions that may arise from many specific characteristics of the different models and applications, but reduce to a common ground in the final analysis. And we impose an additional condition on mechanisms that we call respectfulness. This condition only applies in environments where agents are indifferent among alternatives, and then does not allow any agent to change the consequences for others by means of her actions, while not affecting

[5]A pioneering paper by Shenker (1993) investigated the connections between individual and group strategy-proof non-bossy social choice rules in economic environments. For a recent reference on efficiency in general environments, see Copic (2017).

their own level of satisfaction. But even in those cases where it has bite, it only applies to situations where the agents' preferences change in specific ways, and is thus a relatively mild requirement, that can be satisfied by attractive rules, as we shall see through our examples in Section 4.[6]

The paper proceeds as follows. In the next Section 2 we present the general framework and define the domain restrictions and the type of mechanisms we shall concentrate on. Section 3 contains the general results and their proofs. Section 4 provides examples of applications and ties them in with our general framework. Appendix A and B contain proofs of results presented in Section 2 and 4, respectively.

2 The model

Let $N = \{1, 2, ..., n\}$ be a finite set of *agents* with $n \geqslant 2$ and A be a set of *alternatives*. Let $\widetilde{\mathcal{R}}$ be the set of all complete, reflexive, and transitive binary relation on A. Let $R_i \in \widetilde{\mathcal{R}}$. denote agent i's preferences, and P_i and I_i be the strict and the indifference part of R_i, respectively. For any $x \in A$, $\overline{L}(R_i, x) = \{y \in A : xP_i y\}$ is the *strict lower contour set of* R_i *at* x, $\overline{U}(R_i, x) = \{y \in A : yP_i x\}$ the *strict upper contour set of* R_i *at* x, and $E(R_i, x) = \{y \in A : yI_i x\}$ is the *indifference class* of R_i at x.

It will be useful to pay attention to the relationship between certain pairs of preferences.

Definition 1 *We say that $R'_i \in \widetilde{\mathcal{R}}$ is an x-**monotonic transform of** $R_i \in \widetilde{\mathcal{R}}$ if there exists a set $E_x^{R_i} \subseteq E(R_i, x)$, $x \in E_x^{R_i}$ such that the following conditions hold:*
(i) for any $z \in E_x^{R_i}$, $E_x^{R_i} \subseteq E(R'_i, z)$,
(ii) for any $z \in E_x^{R_i}$, for any $y \in A \backslash E_x^{R_i}$, $[zR_i y \Rightarrow zP'_i y]$.

Equivalently, (ii) can be written in terms of lower contour sets as follows: for any $z \in E_x^{R_i}$, $\overline{L}(R'_i, z) \supseteq L(R_i, z) \backslash E_x^{R_i}$. Note that no constraint is imposed on elements in $\overline{U}(R_i, z)$ with respect to their order with z according to R'_i.

In words: R'_i is an x-monotonic transform of R_i if there exists a subset $E_x^{R_i}$ of x's indifference class in R_i, containing x, such that the relative position of its elements has weakly improved when going from R_i to R'_i.[7]

There is an special case of x-monotonic transforms of preferences that is easy to identify and we want to single out.

Definition 2 *We say that $R'_i \in \widetilde{\mathcal{R}}$ is an x-**reshuffling of** $R_i \in \widetilde{\mathcal{R}}$ if (i) $\overline{L}(R_i, x) = \overline{L}(R'_i, x)$, and (ii) $\overline{U}(R'_i, x) = \overline{U}(R_i, x)$.*

[6] Respectfulness when applied to private values is a distant relative of non-bossiness (Satterthwaite and Sonnenschein, 1981), but much less demanding than this or other similar conditions analyzed in Thomson (2016). It is mostly required to avoid manipulations by one agent that could benefit others while not gaining anything in exchange. This is an analogue condition to the one we use in Barberà, Berga, and Moreno (2016) but requiring here invariance in outcomes instead of indifferences in outcomes.

[7] In our previous paper Barberà, Berga, and Moreno (2012), we present a similar condition but with additional requirements.

In words: the indifference class, the strict upper and strict lower contour set of R_i and R_i' at x coincide. However, no restrictions are imposed on the order of alternatives within the strict upper and lower contour sets.

Observe that if $R_i' \in \widetilde{\mathcal{R}}$ is an x-reshuffling of $R_i \in \widetilde{\mathcal{R}}$, if R_i' is an x-monotonic transform of R_i such that $E_x^{R_i} = E(R_i, x) = E(R_i', z)$, and $\overline{L}(R_i, x) = \overline{L}(R_i', x)$.

Example 1 exhibits a variety of possible monotonic transforms.

Example 1 *Types of monotonic transforms.*
For $A = \{x, y, z, t, w\}$, let's represent preferences by an ordered list from better to worse, with parenthesis in case of indifferences. Then,
(i) going from preference $(xw)(yz)t$ to $wx(yz)t$ is an m-reshuffling (thus, an m-monotonic tranform) for $m \in \{y, z, t\}$. The preference $wx(yz)t$ is also a w-monotonic transform of $(xw)(yz)t$, but not an x-monotonic transform of $(xw)(yz)t$.
(ii) going from preference $(xw)(yz)t$ to $x(yz)wt$ is a y-monotonic transform, but not a y-reshuffling. Note that $x(yz)wt$ is also a z and an x-monotonic transform of $(xw)(yz)t$. However, $x(yz)wt$ is not a w-monotonic transform of $(xw)(yz)t$.
(iii) changing from $(xw)(yz)t$ to $(yxw)zt$ is a y-monotonic transform, but not a y-reshuffling. However, $(yxw)zt$ is not an m-monotonic transform of $(xw)(yz)t$ for $m \in \{x, w, z\}$.

In each application, the preferences of agents may be restricted a priori to satisfy certain conditions. In what follows, we denote by $\mathcal{R}_i$ the set of those preferences that are allowed for individual i.[8]

Each agent $i \in N$ is endowed with a type θ_i belonging to a set Θ_i. Each θ_i includes all the information in the hands of i. We denote by $\Theta = \times_{i \in N}\Theta_i$ the set of type profiles. A *type profile* is an n-tuple $\theta = (\theta_1, ..., \theta_n)$. We will write $\theta = (\theta_C, \theta_{N\setminus C})$ when we want to stress the role of coalition C in N.

Although the information about agent's preferences is already contained in each type profile, we find it useful to have a language that allows us to explicitly differentiate between the overall information contained in the types and the specific information that refers to preferences. This is often achieved in the literature by predicating that agents are endowed with a utility function that depends on the profile of types. Since we work with ordinal preferences, we prefer to formalize this dependence by means of rules of the form $R : \Theta \to \mathcal{R} \equiv \times_{i \in N} \mathcal{R}_i$, $\mathcal{R}_i \subseteq \widetilde{\mathcal{R}}$, that assigns a preference profile to each type profile and that we call *preference functions*.[9] We will refer to $R(\theta)$ as the preference profile induced by θ. $R_i(\theta)$ will stand for the induced preferences of agent i at type profile θ. Notice that the domain of R is a Cartesian product including all possible type profiles, but its range may be a non-Cartesian strict subset of $\times_{i \in N}\mathcal{R}_i$.

Following the standard use, we will call *private values environments* those where each agent's component of the preference function only depends on her type. That is, $R_i(\theta) =$

[8] Preferences may be required to be strict, or additively separable, for example. Notice that $\mathcal{R}_i$ may not be the same for different i's, as in the case of selfish preferences in economies with private goods.

[9] In all our applications, we will show that it is convenient to subdivide an individual's type into two separate parts. One part will stand for all the elements of information that the individuals accumulate, the other will stand for the idiosyncratic functional relation with which each individual will process her available information in order to form her preferences over alternatives. We develop this language in Section 4.

$R_i(\theta_i)$ for each agent $i \in N$ and $\theta \in \Theta$. Otherwise, we are in *interdependent values environments.*

We now introduce an example which adapts in ordinal terms the one proposed by Bergemann and Morris (2005) as their Example 1. We shall use it to illustrate several ideas along this section.

Example 2 *An interdependent values example with a non-Cartesian range.*
Let $N = \{1, 2\}$ and $A = \{a, b, c\}$. Each agent i has two possible types: $\Theta_i = \{\underline{\theta}_i, \overline{\theta}_i\}$. The preference function R is defined in Table 1. We write, in each cell, the preferences of both agents for a given type profile as in Example 1. Observe that agent 2's preferences over b and c depend on agent 1's type: $bP_2(\underline{\theta}_1, \underline{\theta}_2)c$ while $cP_2(\overline{\theta}_1, \underline{\theta}_2)b$, that is, we are in an interdependent values environment.

R	$\underline{\theta}_2$		$\overline{\theta}_2$	
$\underline{\theta}_1$	$R_1(\underline{\theta}_1, \underline{\theta}_2)$ acb	$R_2(\underline{\theta}_1, \underline{\theta}_2)$ $b(ac)$	$R_1(\underline{\theta}_1, \overline{\theta}_2)$ bca	$R_2(\underline{\theta}_1, \overline{\theta}_2)$ $a(bc)$
$\overline{\theta}_1$	$R_1(\overline{\theta}_1, \underline{\theta}_2)$ $c(ab)$	$R_2(\overline{\theta}_1, \underline{\theta}_2)$ $c(ab)$	$R_1(\overline{\theta}_1, \overline{\theta}_2)$ $c(ab)$	$R_2(\overline{\theta}_1, \overline{\theta}_2)$ $c(ab)$

Table 1. Preference function for Example 2.

To show that the range of R is not a Cartesian product, note that $\mathcal{R}_1 = \{acb, bca, c(ab)\}$ and $\mathcal{R}_2 = \{b(ac)), a(bc), c(ab)\}$ but the preference profile $(acb, a(bc))$ is not in the range of the preference function R.

Notice that Bergemann and Morris (2005) do not provide any reason why the preference function is the one they propose. That is not necessary for much of our general analysis. Yet, as we shall see later (Section 4), in many applications, the relevant preference function can be derived from the types in specific manners that clarify its economic meaning.

Our results focus on direct mechanisms. In fact, the properties we discuss are best analyzed with reference to the direct mechanism associated to any general one that might be described in terms of different message spaces and outcome functions.

A *direct mechanism on* Θ is a function $f : \Theta \to A$ such that $f(\theta) \in A$ for each $\theta \in \Theta$. From now on, we drop the term "direct" and refer to mechanisms, without danger of ambiguity.

Notice that, by letting Θ be the domain of f, we implicitly assume that all type profiles within this set are considered to be feasible by the designer.

Our results will focus on the characteristics of the domains on which mechanisms are defined.

We shall now identify two important conditions on domains (Definitions 5 and 6) that may or may not be satisfied by given sets of type profiles. Both conditions start by considering sequences of type profiles that result from changing the type of individual agents, one at a time. These sequences are identified in detail in Definitions 3 and 4.

Let $S = \left\{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\right\} \in \prod_{h=1}^{t_S} \Theta_{i(S,h)}$ be a sequence of individual types of length t_S. Agents may appear in that sequence several times or not at all. $I(S) = (i(S, 1), ..., i(S, t_S))$

is the sequence of agents whose types appear in S and $i(S,h)$ is the agent in position h in S.

Given $\theta \in \Theta$ and $S = \left\{ \theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)} \right\} \in \prod_{h=1}^{t_S} \Theta_{i(S,h)}$, we consider the sequence of type profiles $m^h(\theta, S)$ that results from changing one at a time the types of agents according to S, starting from θ. Formally, $m^h(\theta, S) \in \Theta$ is defined recursively so that $m^0(\theta, S) = \theta$ and for each $h \in \{1, ..., t_S\}$, $m^h(\theta, S) = \left(\left(m^{h-1}(\theta, S) \right)_{N \setminus i(S,h)}, \theta^S_{i(S,h)} \right)$.

Definition 3 *Let $\theta \in \Theta$, $S = \left\{ \theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)} \right\} \in \prod_{h=1}^{t_S} \Theta_{i(S,h)}$ as just defined. We call the sequence of type profiles $\left\{ m^h(\theta, S) \right\}_{h=0}^{t_S}$* ***the passage from*** *θ* ***to*** *θ'* ***through*** *S if $m^{t_S}(\theta, S) = \theta'$ for $\theta' \in \Theta$.*

More informally, we say that θ leads to θ' through S.

Notice that a given passage from θ to θ' through S induces a corresponding sequence of preference profiles, $R^h(\theta, S)$ for $h \in \{0, 1, ..., t_S\}$, where for each agent $i \in N$, $R^h_i(\theta, S) \equiv R_i\left(m^h(\theta, S) \right)$.

We can now establish a condition on the connection between sequences of changes in type profiles and the changes in preferences profiles that they induce.

Definition 4 *Let $x \in A$, $\theta, \theta' \in \Theta$. We will say that the passage from θ to θ' through S is x-**satisfactory** if for each $h \in \{1, ..., t_S\}$, $R^h_{i(S,h)}(\theta, S)$ is an x-monotonic transform of $R^{h-1}_{i(S,h)}(\theta, S)$.*

Notice that in the case of private values the order of individuals in S could be changed and the new sequence would still serve the same purpose. This is because the changes in type of each agent only induce changes in the preferences of this agent. By contrast, the precise order of agents $I(S)$ may be crucial in the case of interdependent values. We say that x is the reference alternative when going from θ to θ'.

We use Example 2 above to illustrate the concept of satisfactory and non-satisfactory passages.

Example 2 (continued) *Satisfactory and non-satisfactory passages.*
Let $x = a$, $\theta = (\underline{\theta}_1, \underline{\theta}_2)$, $\theta' = (\overline{\theta}_1, \underline{\theta}_2)$, and $S = \left\{ \overline{\theta}_2, \overline{\theta}_1, \underline{\theta}_2 \right\}$ a sequence of individual types. Note that, $I(S) = \{2, 1, 2\}$ and $t_S = 3$. We claim that the passage from θ to θ' through S is a-satisfactory. To show it, we have to check that for each $h \in \{1, 2, t_S = 3\}$, $R^h_{i(S,h)}(\theta, S)$ is an a-monotonic transform of $R^{h-1}_{i(S,h)}(\theta, S)$.
For that, observe first that $R^0_{i(S,1)}(\theta, S) = R_2(\underline{\theta}_1, \underline{\theta}_2)$, $R^1_{i(S,1)}(\theta, S) = R_2(\underline{\theta}_1, \overline{\theta}_2)$, $R^1_{i(S,2)}(\theta, S) = R_1(\underline{\theta}_1, \overline{\theta}_2)$, $R^2_{i(S,2)}(\theta, S) = R_1(\overline{\theta}_1, \overline{\theta}_2)$, $R^2_{i(S,2)}(\theta, S) = R_2(\overline{\theta}_1, \overline{\theta}_2)$, and $R^3_{i(S,2)}(\theta, S) = R_2(\overline{\theta}_1, \underline{\theta}_2)$. Then, using the table in Example 2, note that the following three facts hold: $R_2(\underline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_2(\underline{\theta}_1, \underline{\theta}_2)$. Moreover, $R_1(\overline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2)$. Finally, $R_2(\overline{\theta}_1, \underline{\theta}_2)$ is an a-reshuffling of $R_2(\overline{\theta}_1, \overline{\theta}_2)$.
Let $x = a$, $\theta = (\underline{\theta}_1, \underline{\theta}_2)$, $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, and $S = \left\{ \overline{\theta}_1, \overline{\theta}_2 \right\}$ a sequence of individual types. Note that, $I(S) = \{1, 2\}$ and $t_S = 2$. We claim that the passage from θ to θ' through S is

not a-satisfactory. To show it, observe that for $h = 1$, $R^h_{i(S,h)}(\theta, S)$ *is not an a-monotonic transform of* $R^{h-1}_{i(S,h)}(\theta, S)$. *By definition,* $R^1_{i(S,1)}(\theta, S) = R_1(\overline{\theta}_1, \underline{\theta}_2)$, $R^0_{i(S,1)}(\theta, S) = R_1(\theta)$, *and* $R_1(\overline{\theta}_1, \underline{\theta}_2)$ *is not an a-monotonic transform of* $R_1(\theta)$ *since* $\overline{L}(R_1(\overline{\theta}_1, \underline{\theta}_2)) \not\supseteq L(R_1(\theta))$.

Armed with these definitions we now identify the first restriction on sets of types profiles that we are interested in.

Definition 5 *We say that* Θ *is* ***knit*** *if for any two pairs formed by an alternative and a type profile each,* (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$, $\theta \neq \widetilde{\theta}$, *there exist* $\theta' \in \Theta$ *and sequences of types* S *and* $\widetilde{S}$, *such that the passage from* θ *to* θ' *through* S *is* x*-satisfactory and the passage from* $\widetilde{\theta}$ *to* θ' *through* $\widetilde{S}$ *is* z*-satisfactory.*

Notice that in this definition x and z can be the same. Two important remarks are in order. First, whether or not a domain is knit will depend on the preference function through satisfactoriness. Moreover, when going through proofs of knitness (see Remark 1, for example) the reader can observe that for some pairs formed by an alternative and a type profile each, there exist several type profiles and passages that work. Knitness requires only the existence on one such way. Here is an example.

Remark 1 *The domain in Example 2 is knit.*

To check that the domain $\Theta = \left\{(\underline{\theta}_1, \underline{\theta}_2), (\underline{\theta}_1, \overline{\theta}_2), (\overline{\theta}_1, \underline{\theta}_2), (\overline{\theta}_1, \overline{\theta}_2)\right\}$ is knit, we must prove that all pairs of alternatives and types can be connected through satisfactory sequences. To do that, we will show how to choose the appropriate ones for two specific cases, and then argue that all others can be reduced essentially to one of the patterns we shall follow.
<u>Case 1</u>. $(x, \theta) = (a, (\underline{\theta}_1, \underline{\theta}_2))$ and $(z, \widetilde{\theta}) = (b, (\overline{\theta}_1, \underline{\theta}_2))$.
Define $\theta' = \widetilde{\theta} = (\overline{\theta}_1, \underline{\theta}_2)$, $S = \left\{\overline{\theta}_2, \overline{\theta}_1, \underline{\theta}_2\right\}$ (thus, $I(S) = \{2, 1, 2\}$ and $t_S = 3$), $\widetilde{S} = \varnothing$ (thus, $I(\widetilde{S}) = \varnothing$ and $t_{\widetilde{S}} = 0$). Note that since $\theta' = \widetilde{\theta}$, then $\widetilde{\theta}$ trivially leads to θ' through $\widetilde{S}$ and this passage from $\widetilde{\theta}$ to θ' is b-satisfactory. We need to show that θ leads to θ' through S and the passage is a-satisfactory. For that we need to observe using Table 1 that the three (t_S) following facts hold: $R_2(\underline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_2(\underline{\theta}_1, \underline{\theta}_2)$. Moreover, $R_1(\overline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2)$. Finally, $R_2(\overline{\theta}_1, \underline{\theta}_2)$ is an a-reshuffling of $R_2(\overline{\theta}_1, \overline{\theta}_2)$.
<u>Case 2</u>. $(x, \theta) = (c, (\underline{\theta}_1, \underline{\theta}_2))$ and $(z, \widetilde{\theta}) = (a, (\underline{\theta}_1, \overline{\theta}_2))$.
Define $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, $S = \left\{\overline{\theta}_1, \overline{\theta}_2\right\}$ (thus, $I(S) = \{1, 2\}$ and $t_S = 2$), $\widetilde{S} = \left\{\overline{\theta}_1\right\}$ (thus, $I(\widetilde{S}) = \{1\}$ and $t_{\widetilde{S}} = 1$). As above, first we need to show that θ leads to θ' through S and the passage is a-satisfactory. For that we need to observe using Table 1 that the two (t_S) following facts hold: $R_1(\overline{\theta}_1, \underline{\theta}_2)$ is a c-monotonic transform of $R_1(\underline{\theta}_1, \underline{\theta}_2)$. Moreover, $R_2(\overline{\theta}_1, \overline{\theta}_2)$ is a c-reshuffling of $R_2(\overline{\theta}_1, \underline{\theta}_2)$.
Second, we need to show that $\widetilde{\theta}$ leads to θ' through $\widetilde{S}$ and the passage is a-satisfactory. For that we need to observe using the table that $R_1(\overline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2)$.
To finish the proof of knitness we should consider all remaining combinations of (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$. Observe that each one of those cases can be embedded in either Case G1 or

Case G2 below, which generalize Cases 1 and 2, respectively.
Case G1. (x, θ) and $(z, \widetilde{\theta})$ such that $x \in \{a, b\}$.
Case G2. (x, θ) and $(z, \widetilde{\theta})$ such that $x = c$.
To prove knitness for Case G1, consider $\theta' = \widetilde{\theta}$, $\widetilde{S} = \varnothing$, and S will depend on θ and $\widetilde{\theta}$. Similarly, to prove knitness for Case G2, consider $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, $S = \{\overline{\theta}_1, \overline{\theta}_2\}$ (thus, $I(S) = \{1, 2\}$ and $t_S = 2$), and $\widetilde{S}$ will depend on θ and $\widetilde{\theta}$.

Our next restriction is less demanding because it only requires to connect some pairs of type profiles, and only for some pairs of reference alternatives. That is, whether or not a domain is partially knit will depend, as for knitness, on the preference function through satisfactoriness but applied only to certain type profiles and alternatives.

For any $\theta \in \Theta$ and $x, z \in A$, let $C(\theta, z, x) = \{i \in N : zR_i(\theta)x\}$ and $\overline{C}(\theta, z, x) = \{j \in N : zP_j(\theta)x\}$.

Definition 6 *We say that Θ is **partially knit** if for any two pairs formed by an alternative and a type profile each, (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$, $\theta \neq \widetilde{\theta}$, such that $\overline{C}(\theta, z, x) \neq \emptyset$, $\#C(\theta, z, x) \geq 2$, and $\widetilde{\theta}_j = \theta_j$ for any $j \in N \backslash C(\theta, z, x)$, then there exist $\theta' \in \Theta$ and sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.*

Clearly, if a domain is knit it is also partially knit.

Notice that, here again, partial knitness is satisfied as long as there is one satisfactory passage for each relevant pair of alternatives and profiles.

Although the above definitions are general, we want to remark that essentially no domain will be knit in private values environments, as stated in Proposition 1 and proved in Appendix A.

Proposition 1 *In a private values environment, if there exist $\theta_i, \widetilde{\theta}_i \in \Theta_i$ such that $R(\theta_i) \neq R(\widetilde{\theta}_i)$ for some $i \in N$, Θ is not knit.*

If we think of the classical private values framework of social choice theory, where types are identified with preferences, our Proposition 1 implies that the universal domain of strict preferences is not knit. Yet, it is partially knit as shown in the next Proposition 2 (see the proof in Appendix A).

Proposition 2 *The universal domain of strict preferences in the classical social choice problem is partially knit.*

Another interesting well-known example of a partially knit domain in the classical private values framework of social choice is the domain of strict single-peaked preferences on a finite set of alternatives (Moulin, 1980) which is not knit by Proposition 1. See the following Proposition 3 proved in Appendix A.

Proposition 3 *The domain of strict single-peaked preferences on a finite set of alternatives in the classical social choice problem is partially knit.*

The domain of preferences in the classical private values framework of one-to-one matching problem, where preferences are strict over individuals assignments, is also partially knit (but again not knit). See the following Proposition 4 proved in Appendix A.

Proposition 4 *The domain of preferences in the classical one-to-one matching problems is partially knit.*

Until now, we have concentrated on the properties of potential domains on which mechanisms may be defined. We now turn attention to some properties of the mechanisms themselves.

We first look at incentives. *Ex post incentive compatibility* requires, for all agents to prefer truthtelling at a given type profile θ, if all the other agents also report truthfully.[10]

Definition 7 *A mechanism f is **ex post incentive compatible** on Θ if, for all agent $i \in N$, $\theta \in \Theta$, and $\theta'_i \in \Theta_i$, $f(\theta)R_i(\theta)f(\theta'_i, \theta_{N\backslash\{i\}})$.*

We say that an agent $i \in N$ *can ex post profitably deviate under mechanism f at $\theta \in \Theta$* if there exists $\theta'_i \in \Theta_i$ such that $f(\theta'_i, \theta_{N\backslash\{i\}})P_i(\theta)f(\theta)$. Note that ex post incentive compatibility requires that no agent can profitably deviate at any type profile.

Another form of profitable deviations is by means of coalitions. *Ex post group incentive compatibility* requires, for all coalition of agents, each member to prefer truthtelling at a given type profile θ, if all the other agents outside the coalition also report truthfully.

Definition 8 *A mechanism f is **ex post group incentive compatible** on Θ if, for all coalition $C \subseteq N$, $\theta \in \Theta$, $\theta'_C \in \times_{i\in C}\Theta_i$, and $i \in C$, $f(\theta)R_i(\theta)f(\theta'_C, \theta_{N\backslash C})$.*

We say that *a coalition $C \subseteq N$ can ex post profitably deviate under mechanism f at $\theta \in \Theta$* if there exists $\theta'_C \in \times_{i\in C}\Theta_i$ such that for all agent $i \in C$, $f(\theta'_C, \theta_{N\backslash C})R_i(\theta)f(\theta)$ and for some $j \in C$, $f(\theta'_C, \theta_{N\backslash C})P_j(\theta)f(\theta)$. Note that ex post group incentive compatibility requires that no coalition of agents can profitably deviate at any type profile.[11]

Finally, we shall require our mechanisms to satisfy a condition that we call *respectfulness.* It is a relatively weak requirement since it only applies to some limited changes in type profiles, and has no bite in some important cases (for example, in public good economies where agents' preferences are strict). The condition essentially requires that for those limited changes in type profiles, no agent can affect the outcome (for her and for others) unless she changes her level of satisfaction.

Definition 9 *A mechanism f is **(outcome) respectful** on Θ if*

$$f(\theta)I_i(\theta)f(\theta'_i, \theta_{N\backslash\{i\}}) \text{ implies } f(\theta) = f(\theta'_i, \theta_{N\backslash\{i\}}),$$

for each $i \in N$, $\theta \in \Theta$, and $\theta'_i \in \Theta_i$ such that $R_i(\theta'_i, \theta_{N\backslash\{i\}})$ is a $f(\theta)$-monotonic transform of $R_i(\theta)$.

[10] This property is called uniform incentive compatibility by Holmstrom and Myerson (1983). See also Bergemann and Morris (2005).

[11] Notice that we allow for some agents to participate in the profitable deviation without strictly gaining from it. Moreover, we also allow for some agents not to change their types. That facilitates the deviation by groups and therefore makes our concept of ex post group incentive compatibility to be strong.

For short, we call this condition respectfulness. An additional classical requirement that a mechanism may or may not satisfy is that of *Pareto efficiency*.

Definition 10 *A mechanism f is* ***Pareto efficient*** *on Θ if for all $\theta \in \Theta$, there is no alternative x in the range of f such that $xR_i(\theta)f(\theta)$ for all $i \in N$ and $xP_j(\theta)f(\theta)$ for some $j \in N$.*[12]

Notice that ex post group incentive compatibility implies Pareto efficiency, since otherwise the grand coalition could profitably deviate.

3 The results

Our first result shows that only constant mechanisms can be ex post incentive compatible and respectful on knit domains. Before we prove the theorem, let's comment on its importance and implications. The conclusion of Theorem 1 is very strong, and it is in the same vein than the one in Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006) obtain under completely different premises. In our case, the emphasis is on the relevance of the domain on which the mechanism has to be defined. The theorem also restricts attention to mechanisms that are respectful, but notice that this requirement does not always have bite. It is irrelevant when the preferences of all agents under all type profiles are strict. Also observe that since we work with functions, our environments are separable, in the sense of Bergemann and Morris (2005) who also show (see their Proposition 2) that in this case only rules that are ex post incentive compatible could be interim incentive compatible. Therefore, our theorem also applies for the latter weaker requirement, whatever the priors of agents might be, and with no need to be specific about them.

Theorem 1 *Let $f : \Theta \to A$ be a mechanism. If Θ is knit and f is ex post incentive compatible and respectful on Θ, then f is constant.*

Proof. Let Θ be a knit set of type profiles and let f be a ex post incentive compatible and respectful mechanism on Θ. Assume, by contradiction, that f was not constant. Then, there will be $x, z \in A$, $x \neq z$ such that $x = f(\theta)$ and $z = f(\widetilde{\theta})$ for some θ and $\widetilde{\theta}$ in Θ. Since Θ is knit, for the two pairs formed by an alternative and a type profile, (x, θ) and $(z, \widetilde{\theta}) \in A \times \Theta$, there exist $\theta' \in \Theta$ and two sequences $S = \{\theta_{i(S,1)}, ..., \theta_{i(S,t_S)}\}$, $\widetilde{S} = \{\widetilde{\theta}_{i(\widetilde{S},1)}, ..., \widetilde{\theta}_{i(\widetilde{S},t_{\widetilde{S}})}\}$ such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Now, we will show the following:

(a) for each $h \in \{1, ..., t_S\}$, $f(m^h(\theta, S)) = x$, and

(b) for each $h \in \{1, ..., t_{\widetilde{S}}\}$, $f(m^h(\widetilde{\theta}, \widetilde{S})) = z$.

Statements in (a) and (b) yield to a contradiction. By definition of the sequences S and $\widetilde{S}$, we know that $m^{t_S}(\theta, S) = m^{t_{\widetilde{S}}}(\widetilde{\theta}, \widetilde{S}) = \theta'$. However, $f(\theta') = f(m^{t_S}(\theta, S)) = x$ by (a) while

[12]Note that our Pareto efficiency is on the range of the mechanism, but it collapses to Pareto efficiency when the range of the mechanim is the set of alternatives.

$f(\theta') = f(m^{t_{\widetilde{S}}}(\widetilde{\theta}, \widetilde{S})) = z$ by (b).
We prove (a) in steps, from $h = 1$ to $h = t_S$. The proof of (b) is identical and omitted.
Step 1. Let $h = 1$. By Definition 4, $R^1_{i(S,1)}(\theta, S)$ is an x-monotonic transform of $R^0_{i(S,1)}(\theta, S) = R_{i(S,1)}(\theta)$. (1)
Observe that $f(m^1(\theta, S)) \notin \overline{U}\left(R^1_{i(S,1)}(\theta, S), x\right)$. (2)
(otherwise, if $f(m^1(\theta, S)) \in \overline{U}\left(R^1_{i(S,1)}(\theta, S), x\right)$ by (1), $f(m^1(\theta, S))P_{i(S,1)}(\theta)x$ contradicting ex post incentive compatibility since $i(S, 1)$ would profitable deviate under f at θ via $\theta^S_{i(S,1)}$).
Moreover, $f(m^1(\theta, S)) \notin \overline{L}\left(R^1_{i(S,1)}(\theta, S), x\right)$. (3)
(otherwise, if $f(m^1(\theta, S)) \in \overline{L}\left(R^1_{i(S,1)}(\theta, S), x\right)$, we would get a contraction to ex post incentive compatibility since $i(S, 1)$ would profitable deviate under f at θ via $\theta^S_{i(S,1)}$).
By (2) and (3) we have that $f(m^1(\theta, S)) \in E\left(R^1_{i(S,1)}(\theta, S), x\right)$. (4)
Observe that by (1) we obtain that $E\left(R^1_{i(S,1)}(\theta, S), x\right) \subseteq E\left(R^0_{i(S,1)}(\theta, S), x\right)$. (5)
Thus, (4) and (5) show that $f(m^1(\theta, S)) \in E\left(R_{i(S,1)}(\theta), x\right)$. Then, by respectfulness, we get that $f(m^1(\theta, S)) = f(\theta) = x$ which ends the proof of (a) for $h = 1$.
Step $h \in \{2, ..., t_S\}$. By repeating the same argument than in Step 1 on the recursive fact that $f(m^{h-1}(\theta, S)) = x$, we obtain that $f(m^h(\theta, S)) = f(m^{h-1}(\theta, S)) = x$. ■

We now prove our second result, showing the equivalence between ex post individual and group incentive compatibility in partially knit domains. A consequence of the latter is Pareto efficiency, a most desirable property of mechanisms. This result has bite for both private and interdependent values environments.

Theorem 2 *Let f be a respectful mechanism that is defined on a partially knit domain. Then, f is ex post incentive compatible if and only if f is ex post group incentive compatible.*

Before we prove the theorem, let us discuss its content and implications, and propose a corollary.

A consequence of ex post group incentive compatibility is Pareto efficiency on the mechanism's range. Hence, the implications that having a good performance regarding incentives may be compatible with efficiency is an invitation to investigate those cases where this may be a promising possibility. It is true that the equivalence may hold in rather vacuous ways, because there are cases where the only ex post incentive compatible rules lack any interest. But there are other cases where there is a real possibility of making these desiderata compatible in non-trivial ways.

Here are three relevant examples of mechanisms for which the result holds non-trivially in private values environments. One of them is the family of social choice functions defined on single-peaked domains when all agents have strict preferences (see Moulin, 1980 and our Proposition 3). The other case is provided by the top trading cycle mechanism for house allocation (see Shapley and Scarf, 1970 and our Proposition 4). Finally, consider the large class of non-trivial strategy-proof rules on the universal domain that one can define when only two alternatives are at stake (see Barberà, Berga, and Moreno, 2012, Manjunath 2012, and our Proposition 2). In all three cases we are dealing with partially knit private values environments, the mechanisms are individual and group strategy-proof, and by no means trivial.[13] Also remark that for the case where the mechanism has more than two alternatives on the range, only dictatorship is strategy-proof on the universal domain, by the Gibbard-Satterthwaite theorem (see Gibbard, 1973 and Satterthwaite, 1975). This is an example in which our Theorem 2 also applies, since the universal domain is partially knit and dictatorships are group strategy-proof, but we use it here as a warning sign that the implications of Theorem 2, as already explained may or may not be of interest depending on the environments.[14]

We have observed before that partially knit environments include private value cases. Then, the result has a second reading, because it is then the case that ex post incentive compatibility becomes equivalent to strategy-proofness, since each agent i's preferences depend on θ only through θ_i. For the same reason, ex post group incentive compatibility becomes equivalent to group strategy-proofness. These remarks lead us to the following corollary.

Corollary 1 *Let f be a respectful mechanism that is defined on a private values partially knit domain. Then, f is strategy proof if and only if it is group strategy-proof.*

Proof of Theorem 2. Let Θ be a partially knit set of types and let f be a respectful mechanism. By definition, ex post group incentive compatibility implies ex post incentive compatibility. To prove the converse, suppose, by contradiction, that there exist $\theta \in \Theta$, $C \subseteq N$, $\#C \geq 2$, $\widetilde{\theta}_C \in \times_{i \in C}\Theta_i$ such that for any agent $i \in C$, $f(\widetilde{\theta}_C, \theta_{N\setminus C}) R_i(\theta) f(\theta)$ and $f(\widetilde{\theta}_C, \theta_{N\setminus C}) P_j(\theta) f(\theta)$ for some agent $j \in C$. Let $z = f(\widetilde{\theta}_C, \theta_{N\setminus C})$ and $x = f(\theta)$. Note that (i) $z \neq x$, (ii) $\overline{C}(\theta, z, x) \neq \varnothing$, $\#C(\theta, z, x) \geq 2$ since $C \subseteq C(\theta, z, x)$, and (iii) $\widetilde{\theta}_j = \theta_j$ for any $j \in N\setminus C(\theta, z, x)$ again since $C \subseteq C(\theta, z, x)$.
Since Θ is partially knit and conditions in Definition 6 are satisfied, there exist $\theta' \in \Theta$ and two sequences of types $S = \{\theta_{i(S,1)}, ..., \theta_{i(S,t_S)}\}$, $\widetilde{S} = \{\widetilde{\theta}_{i(\widetilde{S},1)}, ..., \widetilde{\theta}_{i(\widetilde{S},t_{\widetilde{S}})}\}$ such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.
Although these sequences are not necessarily the same than the ones we used in the proof of Theorem 1, from this point on, we can use the same reasoning as there, and show that
(a) for each $h \in \{1, ..., t_S\}$, $f(m^h(\theta, S)) = x$, and
(b) for each $h \in \{1, ..., t_{\widetilde{S}}\}$, $f(m^h(\widetilde{\theta}, \widetilde{S})) = z$,
again leading to a contradiction. Adding the arguments we have already used in the proof of Theorem 1 we would complete the one for the present theorem. ■

[13] We say that a mechanism f is group strategy-proof if for any coalition $C \subseteq N$, any $\theta \in \Theta$, any $\theta'_C \in \Theta_C$, $f(\theta_C, \theta'_{N\setminus C}) R_i(\theta) f(\theta')$ for any agent $i \in C$. When the condition is imposed only on singleton coalitions $C = \{i\}$, we say that f is strategy-proof. In words, strategy-proofnes requires that all agents prefer truthtelling at a given type profile θ, whatever all the other agents report.

[14] Let us comment on the connection between our results and the Gibbard-Satterthwaite theorem. There is no contradiction between our result in Theorem 1 that only constant mechanisms are strategy-proof and that of the Gibbard-Satterthwaite theorem, since the universal domain where the latter applies is not knit, as shown in Proposition 1, and thus Theorem 1 does not apply.

4 Applications

In this section we present examples of simple environments where our theorems apply.

These examples are inspired in our reading of several relevant papers in the literature. They are framed in the language we have developed in our paper, and they allow us to clarify several of the points we try to make all along.

Examples 3 and 4 refer to deliberative committees and are inspired by our reading of Austen-Smith and Feddersen (2006), who build on the classical Condorcet jury problem and add the possibility that agents share (true or false) information.

Examples 5 and 6 refer to house allocation problems and are this time inspired by the analysis of Che, Kim, and Kojima (2015), regarding the existence of Pareto effcient and ex-post incentive compatible mechanisms.

Examples 7 and 8 refer to auctions and are inspired by some of the models in Dasgupta and Maskin (2000) and Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006).

Until now we have analyzed our environments in abstract terms, and have not discussed the origin of the preference function, which indicates what is the relevant preference profile associated to each profile of types. We have already remarked, when discussing Example 2, that Bergemann and Morris (2005) propose it without any specific explanation regarding where this function comes from. Indeed, this is common to different general discussions of the issues we address here, and digging on the underlying reasons to predicate a given preference function in immaterial for the validity of our theorems. Notice, however, that whether or not a domain is knit, or partially knit, will depend on the preference function that applies in each case, for environments that are otherwise identical. Hence, it is interesting to know, for each specific application, whether or not the underlying phenomenon we want to model is adequately represented by a specific preference function.

In most applications, authors endow agents with a general utility function[15] that may depend on variables that reflect the agent's type and, in the interdependent values case, on other variables that correspond to the types of the rest of agents. The type of an agent is then given by its utility function, and by the values of the arguments that correspond to its private characteristics. Then, the preference profile that is defined once the profile of everybody's characteristics is used to evaluate the utilities of all agents corresponds to what, in our language, would be the image of our preference function. Our general framework has departed from this specific formulation, since we stick to a purely ordinal framework and avoid the use of utility functions. This has allowed us to define domain restrictions that trascend the details of any specific functional form and avoid questions of representability. Since in this section we want to get closer to well studied issues, we also become specific about the form of preference functions, based on the interpretation of different models. That will allow us to show that the choice of preference functions crucially determines whether a domain of application is knit, or partially knit, and has implications on the possibilities of design.

Hence, from now on, we think of types as a combination of two sorts of elements. On the one hand, agents' types may identify the rules through which they would form their pref-

[15] The use of utility functions that represent the preferences of expected utility maximizers is especially useful, a fortiori, to analyze incentive compatibility notions that involve uncertainty regarding the types.

erences, should they be informed about the whole profile of types. We call them *preference formation rules*. The rest of elements in the type can be treated as a second block, and we call them *signals*.[16]

Formally, a type for agent $i \in N$ can be written as $\theta_i = (b_i, s_i) \in \Theta_i$, where b_i is a function from type profiles to individual i's preferences. If we take those elements as the primitives of the model, we can then induce the relevant preference function for each application as follows. Let B_i be the set of possible preference formation rules for individual i. Let S_i be the set of signals for agent i. In all applications we consider type profiles in $\Theta_i \equiv B_i \times S_i$ and the preference function $R : \Theta \longrightarrow \times_{i \in N} \mathcal{R}_i$ is such that for each $i \in N$, $R_i(\theta) = b_i(s)$ where $s = (s_1, ..., s_n)$. In each of the examples that follow we will be precise about the nature of signals and preference formation rules.[17]

4.1 Deliberative Juries

Example 3. A three-person jury $N = \{1, 2, 3\}$ must decide over two alternatives: whether to acquit (A) or to convict (C) a defendant under a given mechanism. The defendant is either guilty (g) or innocent (i). Each juror j gets a signal $s_j = g$ or $s_j = i$.

Jurors's preferences arise from combining the different signals they obtain from the deliberation, according to their particular preference formation rules. These are of two possible kinds, depending on the agents' tendency to convict in view of their observed signals and of those declared by others. In this example, jurors are either high-biased (h) or low-biased (l). High-biased jurors (h) prefer to convict if and only if all other jurors declare the guilty signal and she has also observed it $(\mathbf{s} = (g, g, g))$, whereas low-biased ones (l) prefer to convict if and only if it has observed the guilty signal or at least one other committee member has declared it $(\mathbf{s} \neq (i, i, i))$.

Let CA denote the preference to convict rather than to acquit and AC be the converse order. Each agent can have two preference formation rules that are defined as follows: (1) $b^h(s) = CA$ if $s = (g, g, g)$ and $b^h(s) = AC$, otherwise, and (2) $b^l(s) = AC$ if $s = (i, i, i)$ and $b^l(s) = CA$, otherwise. Here $B_i = B = \{b^h, b^l\}$ for all agents. The preference function is R is such that for each agent $i \in N$, $R_i((b_i^h, s_i), \theta_{-i}) = b^h(s)$ and $R_i((b_i^l, s_i), \theta_{-i}) = b^l(s)$.

The domain of types in this example is knit. Hence we know by Theorem 1 that it will be impossible to design non-constant, ex post incentive compatible, and respectful mechanisms in such framework.

The proof that the domain is knit is in Proposition 5 in Appendix B. Here we simply provide the reader with some hints on the techniques that we use to check for our domain conditions in this example and subsequent ones.[18]

To check knitness for a particular pair of types and alternatives, (A, θ) and $(C, \widetilde{\theta})$, we must show that there are passages to a third type profile θ' which are A-satisfactory from θ

[16] This is a slight abuse of language, because preference formation rules may not be fixed in some applications, and would also be identified as additional signals received by the agents. But we hope this generates no ambiguity.

[17] An even more general formulation of the preference formation rule would simply require that $R_i(\theta) = b_i(\theta)$, but we do not need to use it in any of our applications.

[18] The reader that finds the following argument to be useful to better understand our condition may also find a similar one for partially knit in the text preceding the proof of Proposition 6 in Appendix B.

and C-satisfactory from $\widetilde{\theta}$, respectively. For notational simplicity, we write h and l to denote b^h and b^l, respectively.

Consider the following three type profiles, $\theta = (\theta_1, \theta_2, \theta_3) = ((l,g),(h,g),(l,i))$, $\widetilde{\theta} = (\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\theta}_3) = ((l,g),(h,g),(l,g))$ and $\theta' = (\theta'_1, \theta'_2, \theta'_3) = ((l,i),(h,i),(l,i))$. The profiles of preferences they induce are shown in Table 2.

$R(\theta) = R((l,g),(h,g),(l,i))$	$R(\widetilde{\theta}) = R((l,g),(h,g),(l,g))$	$R(\theta') = R((l,i),(h,i),(l,i))$
$C \quad A \quad C$	$C \quad C \quad C$	$A \quad A \quad A$
$A \quad C \quad A$	$A \quad A \quad A$	$C \quad C \quad C$

Table 2: Columns indicate agents' preferences induced by θ, $\widetilde{\theta}$, and θ', respectively.

As shown in Table 3, it is possible to sequentially move from θ to θ' by successively changing, one by one, the type of the agents. First agent 1 from (l,g) to (h,i), then agent 2 from (h,g) to (h,i) and finally agent 1 from (h,i) to (l,i). According to our notation, $I(S) = (1,2,1)$. Likewise, as shown in Table 4, we can move from $\widetilde{\theta}$ to θ' by successively changing, one by one, the type of some agents. First agent 1, second agent 3, and then agent 2, all from g to i, while their preference formation rules remaining fixed. That is, $I(\widetilde{S}) = (1,3,2)$. In Table 3, alternative A either does not change its relative position (an A-reshuffling), or improves it (an A-monotonic transform). Similarly, in Table 4, the same requirements are satisfied but this time for alternative C.

$R(\theta) = R((l,g),(h,g),(l,i))$	$R((\mathbf{h},\mathbf{i}),(h,g),(l,i))$	$R((h,i),(h,\mathbf{i}),(l,i))$	$R(\theta') = R((\mathbf{l},i),(h,i),(l,i)$
$C \quad A \quad C$	$A \quad A \quad C$	$A \quad A \quad A$	$A \quad A \quad A$
$A \quad C \quad A$	$C \quad C \quad A$	$C \quad C \quad C$	$C \quad C \quad C$

Table 3: Induced agents' preferences given the specified type changes from θ to θ'.

$R(\widetilde{\theta}) = R((l,g),(h,g),(l,g))$	$R((l,\mathbf{i}),(h,g),(l,g))$	$R((l,i),(h,g),(l,\mathbf{i}))$	$R(\theta') = R((l,i),(h,\mathbf{i}),(l,i)$
$C \quad C \quad C$	$C \quad A \quad C$	$C \quad A \quad C$	$A \quad A \quad A$
$A \quad A \quad A$	$A \quad C \quad A$	$A \quad C \quad A$	$C \quad C \quad C$

Table 4: Induced agents' preferences given the specified type changes from $\widetilde{\theta}$ to θ'.

Example 4. Consider the framework of Example 3 and change agents' preference formation rules as follows. Each juror may now be either unswerving or median. Unswerving jurors (u) prefer to convict if and only if they have observed the guilty sign and have also received such a sign from at least another juror. Median jurors (m) again prefer to convict under the same circumstances but also if they receive two guilty signals from other jurors.

For instance, if juror 1 is unswerving she will prefer to convict if either (g,g,g), (g,g,i), or (g,i,g) but if juror 2 is unswerving she will convict if either (g,g,g), (g,g,i), or (i,g,g)).

Yet being median is the same for both agents, they will prefer to convict if either (g,g,g), (g,g,i), (g,i,g), or (i,g,g).

Each agent can have two preference formation rules that are defined as follows: (1) $b_i^u(s) = CA$ if $s_i = g$ and $s_j = g$ for some $j \neq i$ and $b_i^u(s) = AC$, otherwise; (2) $b_i^m(s) = b^m(s)$ such that $b^m(s) = CA$ if $\#\{i \in N : s_j = g\} \geq 2$ and $b^m(s) = AC$, otherwise. Here $B_i = \{b_i^u, b^m\}$. The preference function is R is such that for each agent $i \in N$, $R_i((b_i^u, s_i), \theta_{-i}) = b_i^u(s)$ and $R_i((b^m, s_i), \theta_{-i}) = b^m(s)$.

This domain is partially knit (see Proposition 6 in Appendix B) but not knit. To show that it is not knit, we present a family of mechanisms, that of quota rules, that are non-constant, respectful, and ex post incentive compatible on Θ.[19]

Let $q \in \{1,2,3\}$. A mechanism f is *voting by quota q* if f chooses C for a type profile θ if and only if at least q agents have induced preferences from θ such that C is preferred to A.[20] Formally, for each type profile $\theta = (b,s) \in \Theta$,

$$f(\theta) = C \text{ if and only if } \#\{i \in N : b_i(s) = CA\} \geq q.$$

In Table 5 below we describe all possible results of voting by quota for different values of q in Example 4. We have four matrices, one for each type of agent 3. In the rows of each matrix we write the four types of agent 1 and in the columns the four types of agent 2. In each cell, we write each agent's best alternative according to their preference at a given type profile, followed by the outcome of a quota mechanism. When two outcomes appear in a cell, the one in the left stands for the outcome of voting by quota 3 and the right one is the outcome for both quota 1 and 2, which in this example are always the same.

Given Table 5, it is easy to check that these rules are ex post incentive compatible. In addition, they also satisfy anonymity.

Now, Theorem 2 will ensure that these and other mechanisms that we may know to be ex post incentive compatible for our example will also be ex post group incentive compatible, since the domain is partially knit.

(b_3^m, i)	(b_2^m, i)	(b_2^m, g)	(b_2^u, i)	(b_2^u, g)
(b_1^m, i)	AAA A	AAA A	AAA A	AAA A
(b_1^m, g)	AAA A	CCC C	AAA A	CCC C
(b_1^u, i)	AAA A	AAA A	AAA A	AAA A
(b_1^u, g)	AAA A	CCC C	AAA A	CCC C
(b_3^u, i)	(b_2^m, i)	(b_2^m, g)	(b_2^u, i)	(b_2^u, g)
(b_1^m, i)	AAA A	AAA A	AAA A	AAA A
(b_1^m, g)	AAA A	CCA A/C	AAA A	CCA A/C
(b_1^u, i)	AAA A	AAA A	AAA A	AAA A
(b_1^u, g)	AAA A	CCA A/C	AAA A	CCA A/C

[19] Note that respectfulness is trivially satisfied in these environments where preferences are strict and alternatives have no private component.

[20] See Austen-Smith and Feddersen (2006) and Barberà and Jackson (2004) for papers where these rules are analized.

(b_3^m, g)	(b_2^m, i)	(b_2^m, g)	(b_2^u, i)	(b_2^u, g)
(b_1^m, i)	AAA A	CCC C	AAA A	CCC C
(b_1^m, g)	CCC C	CCC C	CAC A/C	CCC C
(b_1^u, i)	AAA A	ACC A/C	AAA A	ACC A/C
(b_1^u, g)	CCC C	CCC C	CAC A/C	CCC C
(b_3^u, g)	(b_2^m, i)	(b_2^m, g)	(b_2^u, i)	(b_2^u, g)
(b_1^m, i)	AAA A	CCC C	AAA A	CCC C
(b_1^m, g)	CCC C	CCC C	CAC A/C	CCC C
(b_1^u, i)	AAA A	ACC A/C	AAA A	ACC A/C
(b_1^u, g)	CCC C	CCC C	CAC A/C	CCC C

Table 5. For each type profile, each agent's best alternative and feasible outcome of a voting by quota mechanisms.

4.2 Private goods without money

Example 5. Let $N = \{1, 2\}$ be a set of agents, $O = \{a, c\}$ be a set of objects. Each agent must be assigned one and only one object. Thus, the set of alternatives is $A = \{x = (a, c), z = (c, a)\}$, where the first component refers to the object that agent 1 gets. There is no money in this economy.

The type of each agent i is given by a signal s_i in $[0, 1]$ and a unique preference formation rule b_i, which is defined by using a function $g_i : S \to \mathbb{R}$ in the following way. For any given $s \in S$, the image of $b_i(s)$ is the preference where x is at least as good as z if and only if $g_i(s) \geq 0$.

For each agent $i = 1, 2$, we assume that the function g_i is *increasing* in both signals.[21] The domain in Example 5 is knit (see Proposition 7 in Appendix B). Therefore by Theorem 1 only constant mechanisms can be ex post incentive compatible and respectful in this context.

Example 6. We consider the framework of Example 5, except that we change agents' preference formation rules to be induced by the functions $g_1(s) = \min\left(median\left\{\frac{1}{4}, s_1, s_1, s_2\right\}\right) - \frac{1}{4}$ and $g_2(s) = \min\left(median\left\{\frac{1}{4}, s_2, s_2, s_1\right\}\right) - \frac{1}{4}$. For any given $s \in S$, the image of $b_i(s)$ is the preference where x is at least as good as z if and only if $g_i(s) \geq 0$.

The main difference between this example and the preceding one is that now the functions g_i are not strictly increasing, just weakly.

Like in Example 4 above, the domain of types in this example is partially knit (see Proposition 8 in Appendix B) but not knit. To prove it, consider the following mechanisms.

A mechanism $f_{veto\ x}$ is a *veto rule for x* if for any type profile the outcome is agent 1's best alternative when it is unique, and it is agent 2's best alternative otherwise. Formally, for $\theta \in [0, 1]^2$,

$$f_{veto\ x}(\theta) = \left\{ \begin{array}{c} x = (a, c) \text{ if } \theta \in S_{ca}, \text{ and} \\ z = (c, a) \text{ if } \theta \in S_{aa} \cup S_{ac} \cup S_{cc} \cup S^0 \end{array} \right\}.$$

[21] Che, Kim, and Kojima (2015) also impose the following property which they call the *single-crossing property*: $\frac{\partial u_i(\theta)}{\partial s_i} > \frac{\partial u_j(\theta)}{\partial s_i}$ for any $\theta \in \Theta$. However, as they already mention, this condition is not required for the impossibility result to hold.

In view of Theorem 1 the existence of these non-constant, ex post incentive compatible, and respectful mechanisms implies that the domain of types is no longer knit (in Appendix B we show that veto rules satisfy the three properties). Now, Theorem 2 will ensure that these and other mechanisms that we may know to be ex post incentive compatible for our example will also be ex post group incentive compatible, and therefore, Pareto efficient, since the domain is partially knit.

4.3 Auctions

There is one unit of an indivisible good to be auctioned. Let N be the set of buyers (agents). An alternative in this model tells us which single agent, if any, gets the good and what positive price she pays for it, meaning then that the rest agents do not get the good and pay zero. If no agent gets the good, no one pays anything. Formally, an alternative x is written as $x = (x_1, ...x_n) \in A = (\{0,1\} \times \mathbb{R}_+)^n$, with $x_i = (a_i, p_i)$ where $a_i = 1$ and $p_i > 0$ if and only if agent i gets the good, and $p_l = 0$ for all agents l that do not get it.

We assume that agents' preferences are selfish. Agents only care about whether or not they are awarded the good and, if so, about how much they must pay for it. Threfore, we can define their preferences on the part of the alternative that concerns them and then naturally extend such preferences to alternatives.

The type of each agent i is given by a signal, $s_i \in S_i \subseteq \mathbb{R}$ (where S_i has a minimun), and by a unique preference formation rule b_i. This rule is defined by means of an auxiliary function $g_i : S \to \mathbb{R}$. For any given $s \in S$, $b_i(s)$ is the preference where

(1) $(1, p_i)P_i(s)(1, q_i)$ for all $q_i > p_i$ (agent i strictly prefers paying less than more), and

(2) $(1, g_i(s))I_i(s)(0, 0)$ (agent i is indifferent between not getting the good and paying nothing or receiving the good and paying $g_i(s)$).

Notice that $g_i(s)$ is buyer i's valuation of the good, g_i has a minimum in S, and that the preference relation of i is fully determined once we know which alternative $(1, g_i(s))$ is indifferent to $(0, 0)$.

We assume all along this section that for each agent i, g_i satisfies the following standard condition in the literature: (a) g_i is non-decreasing in s_i.

Example 7. Let us assume that, in addition to condition (a), for any agent i, the evaluation will be the lowest possible if all other agents but i receive the lowest signal. This is formally expressed by condition: (b) $g_i(s) = g_i(\underline{s})$ for s such that $s_j = \underline{s}_j$ for all $j \in N\backslash\{i\}$.[22]

Under conditions (a) and (b), the domain in this example is knit[23] (see Proposition 9 in

[22] An example of a g_i function satisfying these properties is presented by Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006). In our notation, consider the case where $g_i(s) = \beta_i + \alpha \prod_{j \in N} s_j$, $\beta_i \in [0, 1]$, $\alpha > 0$ and the signal space is $S_i = [0, 1]$. Note that by fixing β_i and α, we have a unique preference formation rule for each agent.

[23] Our examples are chosen to illustrate our points, and the readers may want to create additional ones or to use them for comparison with alternative results. Take, for instance, the function $g_i(s) = \max\{s_1, ..., s_n\}$, that is used in Ivanov, Levin, and Niederle (2010), for other purposes. Such auxiliary function g_i satisfies condition (a) but not (b), and it could be used to define a knit (hence, also partially knit) domain. Since

Appendix B). Hence, again by Theorem 1 we know that it will be impossible to design non-constant, ex post incentive compatible and respectful mechanisms in such framework. This negative result parallels those in Examples 3 and 5 above, where Theorem 1 also applies.

One could wonder whether it would be possible to find non-constant mechanims by dropping the requirement of respectfulness. We do not have a full answer to this question, but the answer is negative if we substitute condition (a) by the stronger condition (c) g_i is strictly increasing in s_j for all $j \in N$, and the requirement that the good is always allocated. (See Proposition 10 in Appendix B.)

Now, Example 8 and our subsequent remarks will explore the positive consequences of apparently small changes in the preference formation rules.

Example 8. For simplicity, let $N = \{1, 2\}$, $S_i = \{0, 1\}$ for all $i \in N$ and $l, m, h \in \mathbb{R}_+$ with $0 = l < m < h$. The agent's preference formation rule is defined as in the general framework but will now be based on a different auxiliary function that takes three possible values, low, medium and high.

More formally,

$$g_i(s) = \begin{cases} l & \text{if} \quad s_i = 0 \\ m & \text{if} \quad s_i = 1 \text{ and } s_j = 1, \\ h & \text{if} \quad s_i = 1 \text{ and } s_j = 0. \end{cases}$$

Observe that for each agent i, g_i satisfies (a) and the following condition:
(d) g_i is non-increasing in s_j, for all $j \in N \backslash \{i\}$.

Condition (d), in contrast to the cases encompassed in Proposition 10 and to some cases in Example 7, establishes that the valuation of the good by agent i depends negatively on other agents' signals. Note also that Example 8 does not satisfy condition (c).

Now, we assert that the domain of types in this example is not knit, but is partially knit (see Proposition 11 in Appendix B). Therefore, we can apply Theorem 2 and conclude that any ex post incentive compatible and respectful mechanism on that domain will also be ex post group incentive compatible, and therefore, Pareto efficient.

In view of Theorem 1, to prove that is not knit, it is enough to show that the domain admits a non-constant, ex post incentive compatible, and respectful mechanism. Here is such a mechanism.[24] Let $l < p < m$ and $l < p' < m$. Let $f_{p,p'}$ be such that no agent gets the good if both signals are 0, agent 1 gets the good and pays p if her signal is 1, and agent 2 gets the good and pays p', otherwise. Formally, for $\theta \in \{0, 1\}^2$,

$$f_{p,p'}(\theta) = \left\{ \begin{array}{c} ((0,0),(0,0)) \text{ if } s_1 = s_2 = 0, \\ ((1,p),(0,0)) \text{ if } s_1 = 1, \text{ and} \\ ((0,0),(1,p')) \text{ if } s_1 = 0,\ s_2 = 1. \end{array} \right\}$$

Let us complete the discussion of this and related examples with some additional comments. Example 8 provides a scenario where to apply Theorem 2, which is based on the

our purpose is only to provide some examples, we leave the possibility of constructing new ones based on this g_i to the interested readers.

[24] In Lemma 3 in Appendix B we show that $f_{p,p'}$ is ex post incentive compatible and respectful defined on Θ and g_i's as in Example 8.

assumption that changes in some agent's signal induce reverse effects in the preferences of the different participants in the auction. While we can think of environments and signals where this can be the case, the assumption that prevails in the literature on auctions is that all agents respond in the same direction to changes in some agent's signal. Led by this observation, we offer the reader the following additional remark (that is formally justified in Appendix B).

Remark 2 *If we just modify Example 8 and assume that all agents' preferences respond in the same way positively to changes in signals, we can prove the existence in such environment of a mechanism that is respectful, and individually but not ex post group incentive compatible. Hence, this new specification lead to domains that are not partially knit.*

5 Discussion

In this paper we have emphasized the crucial role of domains of definition in determining whether or not satisfactory ex post incentive compatible mechanisms can be designed.

Our classification of domains is not based on specific assumptions about preferences, or the structure of the space of alternatives, or other considerations that end up determining what combinations of types are admissible in specific applications. Rather, we have extracted from different possible special cases what we think are crucial aspects that distinguish some domains from others. These characteristics refer to how different type profiles are interconnected within a given domain.

Since we model the preferences of agents as binary relations, and conduct our analysis in ordinal terms, we have introduced the notion of a preference formation rule, which is useful for our applications but not essential for the general results.

Our conditions do not refer specifically to the structure of the set of types, or to its dimensionality. Since the distinction between one-dimensional and multidimensional signals is often seen as being determinant for the possibility or impossibility of designing efficient mechanisms with good incentive properties, our results suggest that this criterion, however important, needs not always be determinant.

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6 Appendix A

In this appendix we prove propositions stated in Section 2.

Proof of Proposition 1. Let $i \in N$, $\theta_i, \widetilde{\theta}_i \in \Theta_i$, $\theta_i \neq \widetilde{\theta}_i$ be such that $R(\theta_i) \neq R(\widetilde{\theta}_i)$. There will be a pair of alternatives, say x and z, such that $xP_i(\theta_i)z$ and $zR_i(\widetilde{\theta}_i)x$ (otherwise, for $\theta_i, \widetilde{\theta}_i \in \Theta_i$, $R(\theta_i) = R(\widetilde{\theta}_i)$). To show that the set of types Θ is not knit, we prove that for the two pairs $(x, (\theta_i, \theta_{-i}))$, and $(z, (\widetilde{\theta}_i, \theta_{-i}))$ for some θ_{-i}, there does not exist any θ', S, and $\widetilde{S}$ such that the passage from θ to θ' through S be x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ be z-satisfactory. We prove it by contradiction. Suppose otherwise that there exist θ^*, S^*, $\widetilde{S}^*$, such that the passages $\left\{m^h(\theta, S^*)\right\}_{h=0}^{t_{S^*}}$ and $\left\{m^h(\theta, \widetilde{S}^*)\right\}_{h=0}^{t_{\widetilde{S}^*}}$ from θ to θ^* through S^* and $\widetilde{\theta}$ to θ^* through $\widetilde{S}^*$ are x and z-satisfactory, respectively.
Since we are in a private values environment, changes in the type of agent j never affect the induced preferences of other agents, in particular never affect i's induced preferences if $j \neq i$. Moreover, we know that $xP_i(\theta_i)z$ and $zR_i(\widetilde{\theta}_i)x$. These two observations imply that agent i must belong to $I(S^*) \cup I(\widetilde{S}^*)$. That is, i will appear in at least one of these two sequences. We concentrate on the steps of the passage where agent i changes her type and we show that there is no θ^* compatible with x-satisfactory and z-satisfactory passages from θ to θ^* and from $\widetilde{\theta}$ to θ^*.
Without loss of generality, by the remark just after Definition 4, we can assume that all types of agent i in S^* and $\widetilde{S}^*$ appear in the first positions in these sequences. Let's define $I_{S^*,i} \equiv \{h \in \{1, 2, ..., i_{S^*}\} : I(S^*, h) = i\}$ and $I_{\widetilde{S}^*,i} = \left\{h \in \left\{1, 2, ..., i_{\widetilde{S}^*}\right\} : I(\widetilde{S}^*, h) = i\right\}$.
Take $1 \in I_{S^*,i}$. Since $R_i^1(\theta, S^*)$ is an x-monotonic transform of $R_i(\theta_i)$, we have that $xP_i(m_i^1(\theta, S^*))z$. By repeating the same argument for each $h \in I_{S^*,i}$ we finally obtain that $xP_i(m_i^{i_{S^*}}(\theta, S^*))z$ where $m_i^{i_{S^*}}(\theta, S^*) = \theta_i^*$.
Now, take $1 \in I_{\widetilde{S}^*,i}$. Since $R_i^1(\widetilde{\theta}^*, \widetilde{S}^*)$ is a z-monotonic transform of $R_i(\widetilde{\theta}_i^*)$, we have that $zR_i(m_i^1(\widetilde{\theta}^*, \widetilde{S}^*))x$. By repeating the same argument for each $h \in I_{\widetilde{S}^*,i}$ we finally obtain that $zR_i(m_i^{i_{\widetilde{S}^*}}(\theta, \widetilde{S}^*))z$ where $m_i^{i_{\widetilde{S}^*}}(\theta, \widetilde{S}^*) = \theta_i^*$.
As mentioned above, changes in types of agents different from i will not change agent i's preferences. Thus, we have obtained the desired contradiction. On the one hand that $xP_i(\theta^*)z$ and on the other hand, that $zR_i(\theta^*)x$. ■

Proof of Proposition 2. Two relevant observations: types are preferences, that is, $\theta_i = R_i \in \mathcal{R}_i = \Theta_i$ for each $i \in N$. Moreover, changes in j's preferences do not affect i's preferences if $i \neq j$.
Let $\mathcal{U}$ denote the universal set of strict preferences in the classical social choice problem. Thus, $\mathcal{R}_i = \mathcal{U}$. To check partial knitness, take any (x, R), $(z, \widetilde{R}) \in A \times \mathcal{U}^n$ such that $\overline{C}(R, z, x) = C(R, z, x) \neq \varnothing$, $\#C(R, z, x) \geq 2$, and $\widetilde{R}_j = R_j$ for all $j \in N \backslash C(R, z, x)$. Without loss of generality, let $C(R, z, x) = \{1, 2, ..., c\}$ where c denotes its cardinality. Construct S, $\widetilde{S}$ and R' satisfying the condition in partially knitness.
Before, for each $R_i \in \mathcal{U}$, let R_i^z be the preference obtained by lifting z to the first position and keep the relative position of all other alternatives.

Now, start from R and define $S = \{R_1^z, R_2^z, ..., R_c^z\}$ where $t_S = c$. Note that for each $h \in \{1, ..., c\}$, $R_i^h(R, S) = R_i^z$ is an x-reshuffling of i's previous preferences R_i. Then, $R' = R^c(R, S) = R^z$.
Now, start from $\widetilde{R}$ and define $\widetilde{S} = \left\{\widetilde{R}_1^z, \widetilde{R}_2^z, ..., \widetilde{R}_c^z, R_1^z, R_2^z, ..., R_c^z\right\}$ where $t_{\widetilde{S}} = 2c$. Note that for each $h \in \{1, ..., c\}$, $R_i^h(\widetilde{R}, \widetilde{S}) = \widetilde{R}_i^z$ is a z-monotonic transform or a z-reshuffling (if z was already the top) of i's previous preferences $\widetilde{R}_i$, and for $h \in \{c+1, ..., 2c\}$, $R_i^h(\widetilde{R}, \widetilde{S}) = R_i^z$ is a z-reshuffling of i's previous preferences $\widetilde{R}_i^z$ Then, $R' = R^{2c}(\widetilde{R}, \widetilde{S}) = R^z$.
This ends the proof. ■

Proof of Proposition 3. The same two observations as in the proof of Proposition 2 apply: types are preferences, that is, $\theta_i = R_i \in \mathcal{R}_i = \Theta_i$ for each $i \in N$. Moreover, changes in j's preferences do not affect i's preferences if $i \neq j$.
Let A be a finite and ordered set of alternatives in $\mathbb{R}$, the real line. For all $i \in N$, let $\mathcal{R}_i = \mathcal{S}$ be the set of strict single-peaked preferences on A according to the established real numbers order. Thus, $\mathcal{R} = \mathcal{S}^n$. We introduce some notation: Given $R_j \in \mathcal{S}$, $p(R_j)$ denotes the peak, that is, the best alternative, of R_j in A. Given $R_j \in \mathcal{S}$ and $x \in A$, define $r(R_j, x)$ as the first alternative in $\overline{L}(R_j, x)$ in the opposite side of alternative x with respect to $p(R_j)$.
Another useful observation that we use in this proof is that Definition 1 of x-monotonic transform is equivalent to the following one when agents' preferences are strict: "$R_i' \in \widetilde{\mathcal{R}}$ is an *x-monotonic transform of* $R_i \in \widetilde{\mathcal{R}}$ if for any $y \in A \backslash \{x\}$, $[xP_iy \Rightarrow xP_i'y]$". Equivalently, $\overline{L}(R_j, x) \subseteq \overline{L}(\widetilde{R}_j, x)$, being a x-reshuffling when the equality holds.
To check partial knitness, take any $(x, R), (z, \widetilde{R}) \in A \times \mathcal{R}$ such that $\overline{C}(R, z, x) = C(R, z, x) \neq \varnothing$, $\#C(R, z, x) \geq 2$, and $\widetilde{R}_j = R_j$ for all $j \in N \backslash C(R, z, x)$. Without loss of generality, let $x < z$, which implies that $p(R_j) > x$. Also without loss of generality, let $C(R, z, x) = \{1, 2, ..., c\}$ where c denotes its cardinality. Now define $S = \widetilde{S} = C(R, z, x) = \{1, 2, ..., c\}$ and construct for each agent $j \in \{1, 2, ..., c\}$, R_j' depending on the cases below.
Take any $j \in C(R, z, x)$ and consider the following cases.
<u>Case 1</u>. $\widetilde{R}_j$ is such that $x\widetilde{P}_jz$. Take $R_j' \in \mathcal{S}$ such that $p(R_j') \in [x, z)$, $r(R_j, x) = z$, and $zP_j'y$ for all $y < x$. Notice that such R_j' exists, and the two following set inclusions hold: $\overline{L}(R_j, x) \subseteq \overline{L}(R_j', x)$, $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R_j', z)$. Thus, R_j' is both an x-monotonic transform *of* R_j and a z-monotonic transform of $\widetilde{R}_j$.
<u>Case 2</u>. $\widetilde{R}_j$ is such that $z\widetilde{P}_jx$. Consider several subcases.
<u>Case 2.1</u>. $\overline{L}(R_j, x) \subseteq \overline{L}(\widetilde{R}_j, x)$. Let $R_j' = \widetilde{R}_j$ and observe that R_j' is an x-monotonic transform of R_j (obviously, R_j' is a z-monotonic transform of $\widetilde{R}_j$ since $R_j' = \widetilde{R}_j$).
<u>Case 2.2</u>. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$. We distinguish additional subcases which require different definitions of R_j'.
<u>Case 2.2.1</u>. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R_j, z)$. Let $R_j' = R_j$ and observe that R_j' is an x-monotonic transform of R_j (obviously since $R_j' = R_j$) and R_j' is also a z-monotonic transform of $\widetilde{R}_j$.
<u>Case 2.2.2</u>. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ *and* $\overline{L}(R_j, z) \subsetneqq \overline{L}(\widetilde{R}_j, z)$. Note that this implies that either *(a)* $p(R_j), p(\widetilde{R}_j) \in (x, z)$ or else *(b)* $p(R_j), p(\widetilde{R}_j) > z$.

If *(a)* holds, then let R'_j be such that $p(R'_j) \in \Big[\min\{p(R_j), p(\widetilde{R}_j)\}, \max\{p(R_j), p(\widetilde{R}_j)\}\Big]$, $r(R'_j, x) = r(R_j, x)$ and $r(R'_j, z) \geq r(\widetilde{R}_j, z)$. Note that by definition of single-peakedness, such preference R'_j exists.
If *(b)* holds, then let R'_j be such that $p(R'_j) \in \Big[z, \min\{p(R_j), p(\widetilde{R}_j)\}\Big]$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$. Note that by definition of single-peakedness, such preference R'_j exists.
Then, observe that R'_j defined in *(a)* and *(b)* is both an x-monotonic transform of R_j and a z-monotonic transform of $\widetilde{R}_j$ since $\overline{L}(R_j, x) \subseteq \overline{L}(R'_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R'_j, z)$ hold.
<u>*Case 2.2.3*</u>: $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ and $z \in \Big(\min\{p(R_j), p(\widetilde{R}_j)\}, \max\{p(R_j), p(\widetilde{R}_j)\}\Big)$. Assume that $p(R_j) < z < p(\widetilde{R}_j)$, otherwise, a similar argument would work.
Note that this implies that either *(a)* $r(R_j, x) \in \Big(z, p(\widetilde{R}_j)\Big]$ or *(b)* $r(R_j, x) \in \Big(p(\widetilde{R}_j), r(\widetilde{R}_j, x)\Big)$ holds.
If *(a)* holds, then let R'_j be such that $p(R'_j) \in [z, r(R_j, x))$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$. Note that by definition of single-peakedness, such preference R'_j exists.
If *(b)* holds, then let R'_j be such that $p(R'_j) \in \Big[z, \min\{r(R_j, x), r(\widetilde{R}_j, z)\}\Big)$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$.
Then, observe that R'_j in *(a)* and *(b)* is both an x-monotonic transform of R_j and a z-monotonic transform of $\widetilde{R}_j$ since $\overline{L}(R_j, x) \subseteq \overline{L}(R'_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R'_j, z)$ hold.
Repeating the same argument for each $j \in C(R, z, x)$ would end the proof. ■

Proof of Proposition 4. The proof follows the same argument as the one in Proposition 2 given that there is a universal set of strict preferences over individual assignments and preferences are selfish as in Barberà, Berga, and Moreno (2016). Just note that although preferences over individual assignments are strict, preferences over alternatives allow for indifferences, by selfishness: all alternatives with the same individual assignment are indifferent for such individual agent. Thus, in the case of matching $C(R, z, x) \supseteq \overline{C}(R, z, x)$ holds and R_i^z is the preference obtained by lifting z and also all alternatives with the same individual assignment z_i to the first position and keep the relative position of all other alternatives. ■

7 Appendix B

In this appendix we present some aspects of the applications in Section 4 with more detail to prove knitness or partially knitness of the domains of types defined in Examples 3 to 8. We also state and prove some intermediate results required for the auctions application.

Deliverative juries

Example 3 (continued)

Proposition 5 *In Example 3, Θ is knit.*

Proof. To prove knitness we just need to combine the following two results.
(1) Consider a pair formed by (A, θ) for any $\theta \in \Theta$ where $\theta_j = (b_j, s_j)$ for each $j \in N$. Let $\theta' \in \Theta$ be such that $\theta'_1 = (l, i)$ and $\theta'_j = (h, i)$ for any $j \in N \backslash \{1\}$. We now define the sequence S to sequentially go from type profile θ to type profile θ' by successively changing the type of the agents in S satisfying A-satisfactoriness. First change, one by one and in any order, agents' signals from to $s_j \neq i$ to i. By definition of b^l and b^h, in each of the above changes, the induced preferences of the agent changing her type are either an A-reshuffling or an A-monotonic transform of her previous preferences.
Observe that by definition of the preference formation rules b^l and b^h, the following condition is satisfied: if $\widehat{s}_j = i$ for all $j \in N$, all jurors prefer A to C for any $\widehat{b}_j \in B$.
We now change, one by one and in any order, each agent's preference formation rule from $b_j \neq h$ to h for any $j \in N \backslash \{1\}$ and from $b_1 \neq l$ to l in the case of agent 1. By the observation made just above, in each of these changes, the induced preferences of each agent is the same and therefore they are an A-reshuffling of their previous preferences. Then, we have defined S such that θ leads to θ' through S and the passage from θ to θ' is A-satisfactory.
(2) Consider a pair (C, θ) for any $\theta \in \Theta$ where $\theta_j = (b_j, s_j)$ for each $j \in N$. We now define the sequence S to sequentially go from type profile θ to type profile θ' above by successively changing the type of the agents in S satisfying C-satisfactoriness. First change, one by one and in any order, agents from to $s_j \neq g$ to g. By definition of b^l and b^h, in each of the above changes, the induced preferences of the agent changing her type are either a C-reshuffling or a C-monotonic transform of her previous preferences.
Observe that by definition of the preference formation rules b^l and b^h, the following property is satisfied: if $\widehat{s}_j = g$ for all $j \in N$, all jurors prefer C to A for any $\widehat{b}_j \in B$.
We now change one by one, and in any order, each agent's preference formation rule from $b_j \neq h$ to h for any $j \in N \backslash \{1\}$ and from $b_1 \neq l$ to l in the case of agent 1. By the observation made just above, in each of these changes, the preferences of the agents do not change and therefore they are a C-reshuffling of their previous preferences. We now change first agent 1 from g to i. This implies that the preferences of agent 1 does not change but the preferences of the rest of the agents change from C preferred to A, to A preferred to C given that $b_j = h$ for any $j \in N \backslash \{1\}$. Then, we change the type of the rest of the agents one by one from g to i. In each one of these changes the preferences of the agent changing her preferences are the same: A preferred to C. Therefore, we have constructed a passage from θ to θ' that is C-satisfactory. ■

Example 4 (continued)

Before engaging in the proof that the domain of types in Example 4 is partially knit (see Proposition 6), we show it through a particular example.

For notational simplicity, from now to the end of the example and when no confusion arises, we write u and m to denote b^u and b^m, respectively.

Consider a particular pair of types and alternatives, (A, θ) and $(C, \widetilde{\theta})$ where $\theta = ((u, g), (u, i), (m, g))$ and $\widetilde{\theta} = ((m, i), (u, i), (u, g))$. Let $\theta' = ((m, i), (u, i), (m, g))$. The profiles of preferences they induce are shown in Table 6.

$R(\theta) = R((u,g),(u,i),(m,g))$	$R(\widetilde{\theta}) = R((m,i),(u,i),(u,g))$	$R(\theta') = R((m,i),(u,i),(m,g))$
$C \quad A \quad C$	$A \quad A \quad A$	$A \quad A \quad A$
$A \quad C \quad A$	$C \quad C \quad C$	$C \quad C \quad C$

Table 6: Columns indicate agents' preferences induced by θ, $\widetilde{\theta}$, and θ', respectively.

We can check that $\overline{C}(\theta, C, A) = C(\theta, C, A) = \{1, 3\}$ and $\widetilde{\theta}_2 = \theta_2$ (that is, requirements in Definition 6 are satisfied). As shown in Table 7 below, it is possible to sequentially move from θ to θ' by successively changing, one by one, the type of the agents. In this case, agent 1 from (u, g) to (m, i). According to our notation, $I(S) = (1)$. Likewise, as shown in Table 8 below, we can move from $\widetilde{\theta}$ to θ' by successively changing, one by one, the type of some agents. In this case, agent 3 from (u, g) to (m, g), that is, $I(\widetilde{S}) = (3)$. In Table 7, note that the change in the preference of agent 1 is a A-monotonic transform of the previous preferences that also involves a change in the preference of agent 3. Similarly, in Table 8, notice that the change in the preference of agent 3 is a C-reshuffling of her previous preferences.

$R(\theta) = R((u,g),(u,i),(m,g))$	$R(\theta') = R((\mathbf{m},\mathbf{i}),(u,i),(m,g))$
$C \quad A \quad C$	$A \quad A \quad A$
$A \quad C \quad A$	$C \quad C \quad C$

Table 7: Induced agents' preferences given the specified type changes from θ to θ'.

$R(\widetilde{\theta}) = R((m,i),(u,i),(u,g))$	$R(\theta') = R((m,i),(u,i),(\mathbf{m},g))$
$A \quad A \quad A$	$A \quad A \quad A$
$C \quad C \quad C$	$C \quad C \quad C$

Table 8: Induced agents' preferences given the specified type changes from $\widetilde{\theta}$ to θ'.

In Tables 7 and 8, we have illustrated the idea of partially knitness for two given type profiles. We now show that any relevant pair of type profiles are connected through two appropriate sequences.

Proposition 6 *In Example 4, Θ is partially knit.*

Proof. Take two pairs (A, θ), $(C, \widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta, C, A) = C(\theta, C, A) \neq \varnothing$, $\#C(\theta, C, A) \geq 2$, and for $j \in N \backslash C(\theta, C, A)$, $\widetilde{\theta}_j = \theta_j$. By definition, for all $j \in N$, $\theta_j = (b_j, s_j)$ and $\widetilde{\theta}_j = (\widetilde{b}_j, \widetilde{s}_j)$. We have to show that there exist $\theta' \in \Theta$ and sequences of types S and $\widetilde{S}$ such that θ leads to θ' through S, $\widetilde{\theta}$ leads to θ' through $\widetilde{S}$, and the passages from θ and $\widetilde{\theta}$ to θ' are, respectively, A and C-satisfactory.
Let $\theta' \in \Theta$ be such that $\theta'_j = (b_j, g)$ for any $j \in C(\theta, C, A)$ and $\theta'_j = \theta_j$ for any $j \in N \backslash C(\theta, C, A)$. Define the sequence $S = \{(b_k, g)\}$, where $k \in C(\theta, C, A)$ and $s_k = i$. Note that $\#I(S) \leq 1$.

By definition of the preference formation rules in the example, if some agent j prefers C to A, the signal profile must be such that at most one agent k has signal i: $s_k = i$. Thus, S is well-defined. Moreover, $b_k = b^m$ since for unswerving jurors to have C over A their signal must be g. And by definition of b^m increasing the support for g implies that preferences remain C over A for agents k and will be C over A for the other agents.

Therefore, we have defined S to go from θ to θ' through S and the passage is A-satisfactory. We now sequentially go from $\widetilde{\theta}$ to θ' by successively changing the type of the agents in $C(\theta, C, A)$, one by one in any order, from to $\widetilde{s}_j \neq g$ to g. This set of agents will be $\widetilde{S}$.

By definition of agents' preference formation rules, if one agent changes her signal increasing the support for guilty, then each agents' induced preferences remain either the same as before or change in favor of C. Thus, in each one of the above changes, the induced preference of the agent changing her type is either a C-reshuffling or a C-monotonic transform of her previous ones.

Now, take any two pairs (C, θ), $(A, \widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta, A, C) = C(\theta, A, C) \neq \varnothing$, $\#C(\theta, A, C) \geq 2$, and for $j \in N \backslash C(\theta, A, C)$, $\widetilde{\theta}_j = \theta_j$, a similar argument would work but defining $\theta' \in \Theta$ to be such that $\theta'_j = (b_j, i)$ for any $j \in C(\theta, A, C)$ and $\theta'_j = \theta_j$ for any $j \in N \backslash C(\theta, A, C)$. Define the sequence $S = \{(b_k, i)\}$, where $k \in C(\theta, A, C)$ and $s_k = g$. Note that $\#I(S) \leq 1$.

By definition of the preference formation rules in the example, if some agent j prefers A to C, the signal profile must be such that only one single agent has signal g or at most two agents k, k' has signal g. In the latter, these two agents are not j, and k, k' have preferences C over A. Thus, S is well-defined. Moreover, by definition of b^m and b^u increasing if the single agent with signal g says i, that preferences of this agent and those of all other agents will be A over C.

Therefore, we have defined S to go from θ to θ' through S and the passage is A-satisfactory. We now sequentially go from $\widetilde{\theta}$ to θ' by successively changing the type of the agents in $C(\theta, A, C)$, one by one in any order, from to $\widetilde{s}_j \neq i$ to i. This set of agents will be $\widetilde{S}$.

By definition of agents' preference formation rules, if one agent changes her signal by increasing the support for innocent, then each agents' induced preferences remain either the same as before or change in favor of A. Thus, in each one of the above changes, the induced preference of the agent changing her type is either a A-reshuffling or a A-monotonic transform of her previous ones. ■

Private goods without money

Example 5 (continued)

We shall prove that the set of type profiles in this example is knit. Before that, observe first that since each agent has a unique preference formation rule that coincides with the preference function, from now till the end of this application, we identify type profiles with profiles of signals and use both words interchangeably. Second, we introduce a partition of the signal space and a useful graphical representation of it which is similar to the one defined in Che, Kim, and Kojima (2015).

Let $\{S_{ac}, S_{ca}, S_{aa}, S_{cc}, S^0\}$ be the partition of S where:
S^0 is the set of signal profiles for which both agents are indifferent between a and c,
S_{ac} is the set of signal profiles for which agent 1 prefers a to c, agent 2 prefers c to a, and

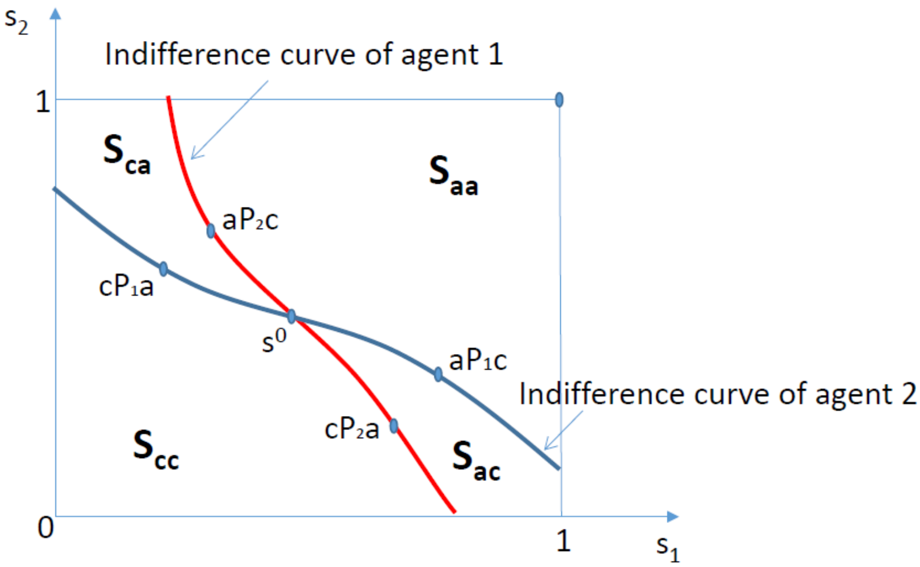


Figure 1. Examples of the partition of S in Example 5.

the preference is strict for at least one agent,
S_{ca} is equally defined after changing the roles of c and a,
S_{aa} is the set of signal profiles for which both agents prefer a to c, and
S_{cc} is equally defined after changing the roles of c and a.

In terms of alternatives, when the signals are in S_{ac} both agents prefer x to z, when they are in S_{ca} both prefer z to x, in S_{aa}, 1 prefers x over z and 2 prefers z over x, in S_{cc}, 1 prefers z over x and 2 prefers x over z, and in S^0 both are indifferent between x and z.

In Example 5, we assume that the sets S_{ac} and S_{ca} are non-empty.

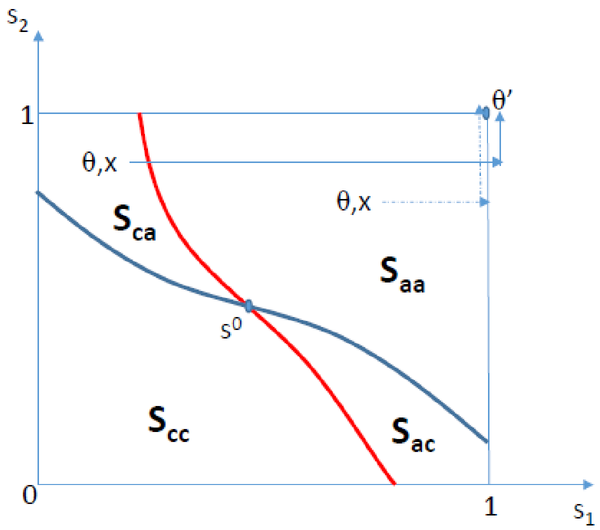
Figure 1 provides a generic representation of these sets whose frontiers correspond to the pairs of signals leading to agents' indifference curves over alternatives: $\{s \in [0,1]^2 : xI_i(\theta)y\}$. Since we have assumed that g_i is increasing in both signals, agents' indifference curves are strictly decreasing and since S_{ac} and S_{ca} are non-empty the two curves will have an interior intersection.[25]

We can now state and prove Proposition 7.

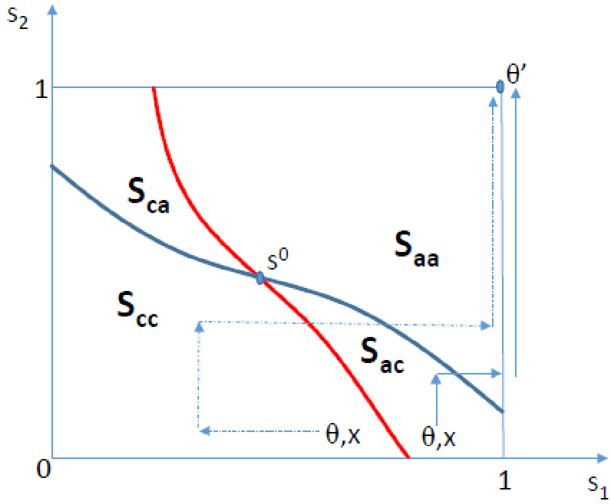
Proposition 7 *In Example 5, Θ is knit.*

Proof. Given any two pairs (x,θ), $(z,\widetilde{\theta}) \in A \times \Theta$ we will show that there exist θ', S, $\widetilde{S}$ such that θ leads to θ' through S, $\widetilde{\theta}$ leads to θ' through $\widetilde{S}$ and the passages are x and z-satisfactory. We choose $\theta' = (1,1)$ independently of the two chosen pairs (x,θ), $(z,\widetilde{\theta}) \in A \times \Theta$. In defining the sequence S from θ to θ' with x as reference alternative, we distinguish two cases where we will end up analyzing all possible $\theta \in \Theta$, in particular $\widetilde{\theta}$.
Case 1. $\theta \in S_{ca} \cup S_{aa} \cup S^0$. First change the type of agent 1 from $\theta_1 \neq 1$ to 1. Since the function g_1 is increasing in type 1, the preferences of agent 1 induced by this change

[25] Although in all pictures corresponding to this example the indifference curves only intersect once, our formal arguments apply to the multiple intersection case.



Picture 2.a. Case 1: θ ∈ S$_{ca}$US$_{aa}$Us^0



Picture 2.b. Case 2: θ ∈ S$_{ac}$US$_{cc}$

Figure 2. Changes of agents' types in Cases 1 and 2, proof of Proposition 7.

are either an x-reshuffling (if $\theta \in S_{aa}$) or an x-monotonic transform ($\theta \in S_{ca} \cup S^0$) of her original ones. Then change the type of agent 2 from θ_2 to 1. Again, since the function g_2 is increasing in type 2, the preferences of agent 2 induced by this change are an x-reshuffling of her original ones (see Picture 2.a in Figure 2).

Case 2. $\theta \in S_{ac} \cup S_{cc}$. In this case we may not be able to change types of agents from $\theta_i \neq 1$ to $(1,1)$ as directly as above.

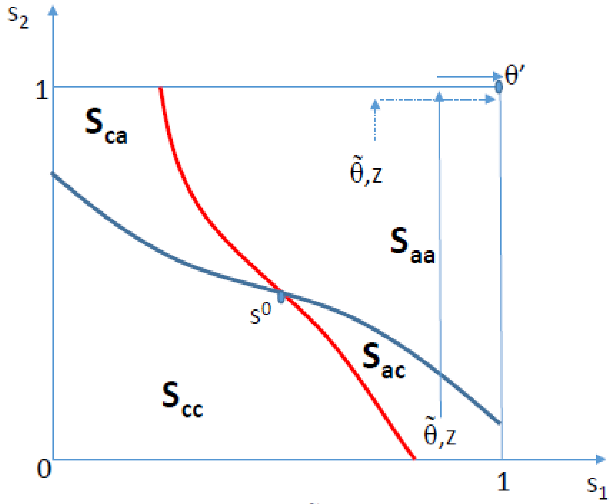
If θ is a type profile from which we could reach another one in S_{aa} by letting the type of the first agent to be 1, we use the same argument as in Case 1: first change the type of agent 1 from $\theta_1 \neq 1$ to 1. The preference of agent 1 induced by this change are either an x-reshuffling (if $\theta \in S_{ac}$) or an x-monotonic transform (if $\theta \in S_{cc}$) of her original ones. Then change the type of agent 2 from θ_2 to 1. The preferences of agent 2 induced by this change are an x-reshuffling of her original ones.

If not, before reaching this situation, the sequence S must start by previous changes of signals, at most one for each agent, as shown in Picture 2.b in Figure 2, that keep us within the element of the partition where θ belongs to. The induced preferences resulting from these previous type changes remain unchanged.

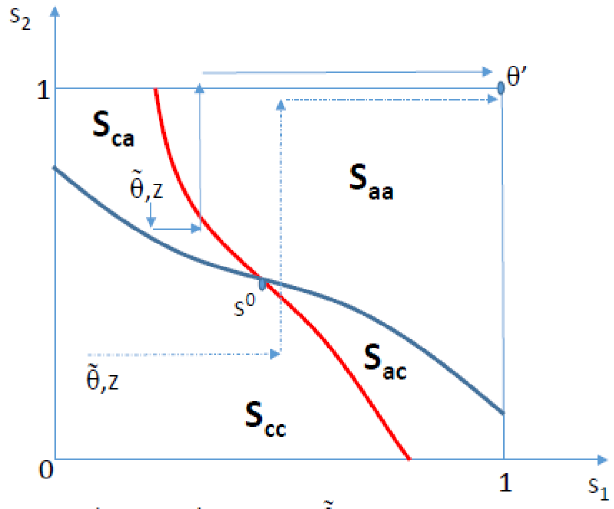
To define the sequence $\widetilde{S}$ from $\widetilde{\theta}$ to θ' with z as reference alternative, we would follow a parallel construction to Cases 1 and 2 above. The relevant cases would now be Case 3: $\widetilde{\theta} \in S_{ac} \cup S_{aa} \cup S^0$ and Case 4: $\widetilde{\theta} \in S_{ca} \cup S_{cc}$ where we would consider all possible type profiles $\widetilde{\theta} \in \Theta$ including θ. The proof for the existence of the sequence $\widetilde{S}$ would require a similar argument to those of Cases 1 and 2, respectively, but changing first agent 2's signal to 1 when required to get to S_{aa}. See the graphical representation in Figure 3.

The construction of these passages shows that our domain is knit as we wanted to show.

■



Picture 3.a. Case 3: $\tilde{\theta} \in S_{ac} \cup S_{aa} \cup s^0$



Picture 3.b. Case 4: $\tilde{\theta} \in S_{ca} \cup S_{cc}$

Figure 3. Changes of agents' types in Cases 3 and 4, proof of Proposition 7.

Example 6 (continued)

Before engaging in the proof that the domain of types in Example 6 is partially knit, observe that the changes in g_i's functions imply that the sets $\overline{S}_{ca} = \{s \in S : zP_1x \text{ and } zP_2x\}$ and $\overline{S}_{ac} = \{s \in S : xP_1z \text{ and } xP_2z\}$ are empty, and that S^0 is not a singleton. Due to the specific form of g_i the indifference set is L-shaped and thick, as shown in Figure 4.

Proposition 8 *In Example 6, Θ is partially knit.*

Proof. As we previously mentioned, two type profiles with the same signal profile are identical. Thus, we identify s with θ. Take any two pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta, z, x) \neq \varnothing$ and $\#C(\theta, z, x) \geq 2$. These two conditions on θ imply that we must only consider $\theta \in S_{ca}$, i.e. where agent 1 strictly prefers z to x and agent 2 is indifferent between x and z. Define $\theta' = \widetilde{\theta}$.

We have to define S such that θ leads to $\theta' = \widetilde{\theta}$ through S and the passage is x-satisfactory. We distinguish two cases. See the graphical representation of both cases in Figure 5.

<u>Case 1</u>. $\widetilde{\theta} \in S_{aa} \cup S_{ca}$. Define $S = \{\widetilde{\theta}_1, \widetilde{\theta}_2\}$ and $I(S) = \{1, 2\}$. Note that if $\widetilde{\theta}, \theta \in S_{ca}$ the proof is obvious since we move along the same set S_{ca} and no agent preferences change.

Suppose that $\widetilde{\theta} \in S_{aa}$. We first increase the signal of agent 1 to $\theta_1' = \widetilde{\theta}_1$. The induced preferences of agent 1 are an x-monotonic transform of her previous ones. Agent 2 turns to strictly prefer z to x, that is, $zR_2(\theta_1', \theta_2)x$. Decrease or increase now agent 2's signal to $\theta_2' = \widetilde{\theta}_2$. Note that agent 2's induced preferences are identical to her previous ones, thus, are obviously an x-reshuffling of the previous ones. So we have gone from θ to θ' through adequate types changes with respect to x.

<u>Case 2</u>. $\widetilde{\theta} \in S_{cc} \cup S_{ac}$. Define $S = \{\widetilde{\theta}_2, \widetilde{\theta}_1\}$ and $I(S) = \{2, 1\}$. We first decrease the signal of agent 2 to $\theta_2' = \widetilde{\theta}_2$. The induced preferences of agent 2 are an x-monotonic transform of her previous ones $R_2(\theta)$ (since $zP_2(\theta)x$ while $xP_2(\theta_1, \theta_2')z$). Agent 1 turns to have the same

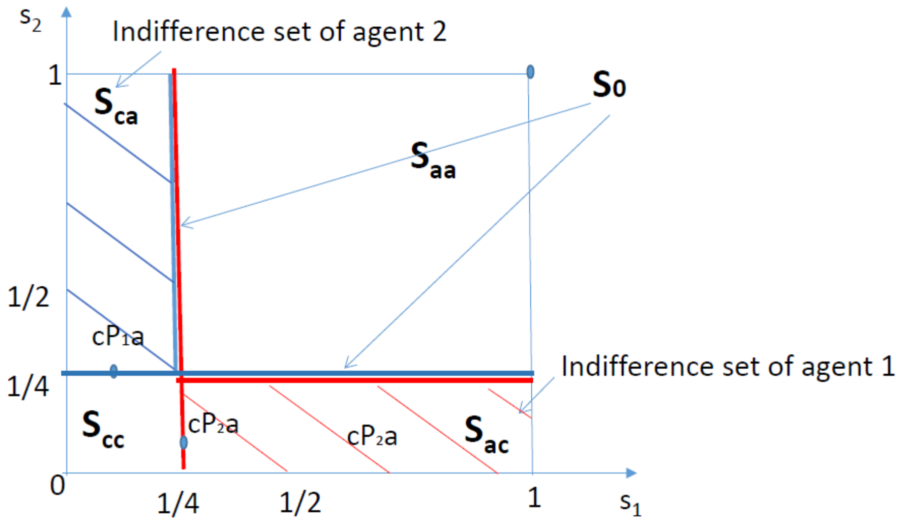


Figure 4. Partition of S in Example 6.

preferences as before, that is, $zR_1(\theta_1, \theta'_2)x$. Now, we decrease or increase agent 1's signal to $\theta'_1 = \widetilde{\theta}_1$. Note that agent 2's induced preferences are either identical to her previous ones (thus, obviously an x-reshuffling of those) or they an x-monotonic transform of $R_1(\theta_1, \theta'_2)$ (since $zP_1(\theta_1, \theta'_2)x$ while $zI_1(\theta'_1)x)$. So, we have gone from θ to θ' through adequate types changes with respect to x.
This ends the proof.
It would remain to consider any two pairs where (z, θ), $(x, \widetilde{\theta}) \in A \times \Theta$ such that such that $\overline{C}(\theta, x, z) \neq \varnothing$ and $\#C(\theta, x, z) \geq 2$, however, a symmetric and similar argument would work.

■

Finally, we show that the mechanism $f_{veto\ x}$ defined in Section 4.3 is non-constant, satisfies ex post incentive compatibility and respectfulness on Θ which shows, by Theorem 1, that the types domain in Example 6 is not knit.

Observe that, by definition, $f_{veto\ x}$ is non-constant and no agent can gain by changing her individual types since they either obtain the same or an indifferent outcome while deviating, or they obtain their best outcome when truthtelling. Ex post group incentive compatibility is straightforward since changing both types it is impossible to weakly improve both agents and strictly one: Note that either agent 1 or 2 strictly lose (we need to check 6 cases: $\theta \in S_{aa}$ and $\theta' \in S_{ca}$ or viceversa; $\theta \in S_{ac}$ and $\theta' \in S_{ca}$ or viceversa; and $\theta \in S_{cc}$ and $\theta' \in S_{ac}$ or viceversa). To show that $f_{veto\ x}$ is respecful, note that the only way for agent 1 to remain indifferent according to her initial preferences $R_1(\theta)$ and get a different outcome when changing her type is when $\theta \in S_{ac}$ and $\theta'_1 < \frac{1}{4}$ such that $(\theta'_1, \theta_2) \in S_{cc}$. However, $R_1(\theta'_1, \theta_2)$ is not an $x = f_{veto\ x}(\theta)$-monotonic transform of $R_1(\theta)$. Similarly, for agent 2, to remain indifferent and get a different outcome when changing her type $\theta \in S^0$ and $\theta_2 \geq \frac{1}{4}$, $\theta'_2 < \frac{1}{4}$. However, $R_2(\theta_1, \theta'_2)$ is not a $z = f_{veto\ x}(\theta)$-monotonic transform of $R_2(\theta)$.

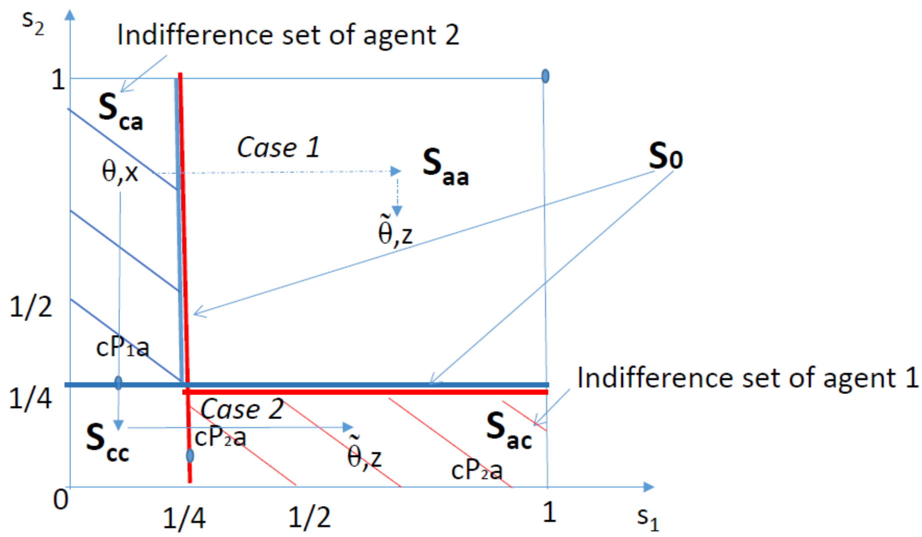


Figure 5. Changes of agents' types, proof of Proposition 8.

Auctions

Example 7 (continued)

Since each agent's preference formation rule is fixed, two type profiles with the same signal profile are identical. Hence we shall use along this section θ and s, indistinctly. The following Lemma is used in the proofs of Propositions 9 and 11 below.

Lemma 1 *Let g_k be non-decreasing in s_k. For all $s \in \Theta$, $R_k(s'_k, s_{-k})$ is a y-monotonic transform of $R_k(s)$ for all $s'_k < s_k$, $k \in N$ and $y \in A$ such that $y_k = (0,0)$.*

Proof. Take $s \in \Theta$, $k \in N$ and $y \in A$ such that $y_k = (0,0)$ and $s'_k < s_k$. Since g_k is non-decreasing in s_k, $g_k(s'_k, s_{N\setminus\{k\}}) \leq g_k(s)$ which means that agent k values the good in signal profile $(s'_k, s_{N\setminus\{k\}})$ at most as under profile s. Thus, $(0,0)$ weakly improves its position in $R_k(s'_k, s_{N\setminus\{k\}})$ compared to its position in $R_k(s)$. Formally, $R_k(s'_k, s_{N\setminus\{k\}})$ is a y-monotonic transform of $R_k(s)$. ■

Proposition 9 *If Θ is such that for each $i \in N$, g_i is as described in Example 7, then Θ is knit.*

Proof. Take any two pairs (x,θ), $(z,\widetilde{\theta}) \in A \times \Theta$. We must find θ', sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Consider $\theta' = \left(\widetilde{s}_i, \underline{s}_{N\setminus\{i\}}\right)$. We first propose a sequence of types $S = \underline{s}$ $(t_S = n)$ with $I(S)$ defined as follows. We initially change, one by one, the signal of agents that do not get the good in x from s_k to $\underline{s}_k$ following the order of natural numbers. If there is one agent i left who was getting the good in x change her signal from s_i to $\underline{s}_i$. In each step $h \in \{1, ..., n-1\}$, by Lemma 1, we obtain that $R_h(m_h(\theta, S))$ is an x-monotonic transform of $R_h(m_{h-1}(\theta, S))$

since for all agents h they do not get the good in x.
As for the last agent in the sequence, her preferences will not change when her signal goes from s_i to $\underline{s}_i$ due to assumption (b) of function g_i.
This completes our argument that the passage from θ to θ' through S is x-satisfactory.
We could repeat exactly the same argument to show that the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory after replacing the roles of θ by $\widetilde{\theta}$ and x by z.
This would end the proof. ■

Proposition 10 *Any ex post group incentive compatible mechanism that always allocate the good to some agent is constant if all auxiliary functions g_i satisfy (b) and (c).*

Proof. Consider f any ex post group incentive compatible mechanism that always allocates the good to some agent and such that all auxiliary functions g_i satisfy (b) and (c). We show that f is constant.
Take $x = f(\overline{s})$ and without loss of generality suppose that agent 1 gets the good and pays p. We show that f is constant by the following steps.

Step 1. For any $s = (\overline{s}_1, s_2, ...s_n)$, such that $s_j \in \Theta_j$ for $j \in N\backslash\{1\}$, then agent 1 gets the good and pays p.
Take any agent that does not get the good, without loss of generality, say agent 2. Consider $s = (s_2, \overline{s}_{N\backslash\{2\}})$ where $s_2 < \overline{s}_2$. By condition (c) of g_i (that is, g_i is strictly increasing in all s_j, $j \in N$), $g_2(s_2, \overline{s}_{N\backslash\{2\}}) < g_2(\overline{s})$. Ex post incentive compatibility implies that $f_2(s_2, \overline{s}_{N\backslash\{2\}}) = f_2(\overline{s}) = (0,0)$. Take now any other agent $k \in N\backslash\{1,2\}$. By condition (c), for each $k \in N\backslash\{1,2\}$, $g_k(s_2, \overline{s}_{N\backslash\{2\}}) < g_k(\overline{s})$. Thus, by ex post group incentive compatibility, we get that for any $k \in N\backslash\{1,2\}$ $f_k(s_2, \overline{s}_{N\backslash\{2\}}) = f_k(\overline{s}) = (0,0)$. This implies that agent 1 gets the good at $(s_2, \overline{s}_{N\backslash\{2\}})$. Moreover, agent 1 pays the same price p. Otherwise, if $p' < p$, coalition N would profitably deviate from $\overline{s}$ to $(s_2, \overline{s}_{N\backslash\{2\}})$ since $f_1(s_2, \overline{s}_{N\backslash\{2\}}) = (1, p')P_1(\overline{s})f_1(\overline{s}) = (1, p)$. The other way arround if $p' > p$.
Repeating $n-2$ additional times the same argument, one for each agent $j \in N\backslash\{1,2\}$, we obtain that for any $s = (\overline{s}_1, s_2, ...s_n)$, such that $s_j \in \Theta_j$ for $j \in N\backslash\{1\}$, agent 1 gets the good and pays p.

Step 2. For any $s = (s_1, \underline{s}_{N\backslash\{1\}})$, such that $s_1 \in \Theta_1$, then agent 1 gets the good and pays p.
By condition (b) of g_1, $g_1(s_1, \underline{s}_{N\backslash\{1\}}) = g_1(\underline{s})$ for any $s_1 \in \Theta_1$. Thus, for any $s_1 \in \Theta_1$, agent 1's preferences $R_1(s_1, \underline{s}_{N\backslash\{1\}})$ coincide with $R_1(\underline{s})$. In particular, $R_1(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$ coincide with $R_1(\underline{s})$. By ex post incentive compatibility, for all $s_1 \in \Theta_1$, $f_1(\overline{s}_1, \underline{s}_{N\backslash\{1\}})I_1(\overline{s}_1, \underline{s}_{N\backslash\{1\}})f_1(s_1, \underline{s}_{N\backslash\{1\}})$, being $f_1(\overline{s}_1, \underline{s}_{N\backslash\{1\}}) = (1, p)$ by Step 1. If agent 1 gets the good at $(s_1, \underline{s}_{N\backslash\{1\}})$, since $R_1(s_1, \underline{s}_{N\backslash\{1\}})$ coincide with $R_1(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$, the price must be the same. That is, $f_1(s_1, \underline{s}_{N\backslash\{1\}}) = (1, p)$ and then the proof of Step 2 ends. Otherwise, take $l \in N\backslash\{1\}$ such that $f_l(s_1, \underline{s}_{N\backslash\{1\}}) = (1, p_l)$. By condition (c) of g_l, $g_l(s_1, \underline{s}_{N\backslash\{1\}}) < g_l(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$. Note that if $p_l < g_l(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$, coalition $\{1, l\}$ can ex post profitably deviate at $(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$ via $(s_1, \underline{s}_l)$ (agent l would strictly gain while 1 remain indifferent). If $p_l \geq g_l(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$, coalition $\{1, l\}$ can ex post profitably deviate at $(s_1, \underline{s}_{N\backslash\{1\}})$ via $(\overline{s}_1, \underline{s}_l)$ (agent l would strictly gain). Thus, we have shown that agent 1 gets the good and pays p at $(s_1, \underline{s}_{N\backslash\{1\}})$, for any $s_1 \in \Theta_1$.

Step 3. For any $s \in \Theta$ such that $s_1 < \overline{s}_1$ and there exists $l \in N\backslash\{1\}$ such that $s_l > \underline{s}_l$, then

agent 1 gets the good and pays p.
Let $C = \{i \in N\backslash\{1\} : s_i > \underline{s}_i\}$.
First, observe that if agent 1 gets the good at s in Step 3, by ex post incentive compatibility, the price must be p. Otherwise, 1 could ex post profitably deviate at s via $\overline{s}_1$ if $p' > p$, or at $(\overline{s}_1, s_{N\backslash\{1\}})$ via s_1 if $p > p'$. Consider the following two cases for which we obtain a contradiction.

Step 3.1. Agent $k \in N\backslash\{1\}$ gets the good at s and $s_k > \underline{s}_k$.
Take an agent $j \in C\backslash\{k\}$ who does not get the good at s and change her type from s_j to $\underline{s}_j$. If $C\backslash\{k\}$ is empty, we have that $f_k(s_k, \underline{s}_{N\backslash\{k\}}) = (1, p')$ and by applying the same argument as in Step 2 we would get $f_k(\underline{s}) = (1, p')$ which contradicts Step 2 applied to $\underline{s}$. Otherwise, by condition (c) of g_j, $g_j(\underline{s}_j, s_{N\backslash\{j\}}) < g_j(s_j, s_{N\backslash\{j\}})$. By ex post incentive compatibility, $f_j(s) = f_j(\underline{s}_j, s_{N\backslash\{j\}}) = (0, 0)$. Again, if $C\backslash\{k, j\}$ is empty, we have that $f_k(s_k, \underline{s}_{N\backslash\{k\}}) = (1, p')$ and by applying the same argument as in Step 2 we would get $f_k(\underline{s}) = (1, p')$ which contradicts Step 2 applied to $\underline{s}$. Otherwise, take $j' \in C\backslash\{k, j\}$, and by condition (c) of $g_{j'}$, $g_{j'}(\underline{s}_j, s_{N\backslash\{j\}}) < g_{j'}(s_j, s_{N\backslash\{j\}})$. If for any $j' \in N\backslash\{k\}$, $f_{j'}(\underline{s}_j, s_{N\backslash\{j\}}) = f_{j'}(s_j, s_{N\backslash\{j\}}) = (0, 0)$, we obtain that $f_k(\underline{s}_j, s_{N\backslash\{j\}}) = (1, p')$ and we repeat the same argument in Step 3.1 for $l \in C\backslash\{k, j\}$. If for some j', $f_{j'}(\underline{s}_j, s_{N\backslash\{j\}}) \neq (0, 0)$, we would get a contradiction to ex post group incentive compatibility: $\{j, j'\}$ would profitably deviate from $(\underline{s}_j, s_{N\backslash\{j\}})$ via $(s_j, s_{j'})$ if $p' > g_j(\underline{s}_j, s_{N\backslash\{j\}})$ and from $(s_j, s_{N\backslash\{j\}})$ via $(\underline{s}_j, s_{j'})$ if $p' \leq g_j(\underline{s}_j, s_{N\backslash\{j\}})$. Thus, $f_k(\underline{s}_j, s_{N\backslash\{j\}}) = (1, p')$.
Repeating the same argument changing one by one the signal from s_l to $\underline{s}_l$ for each $l \in C\backslash\{k\}$, we obtain that $f_k(s_k, \underline{s}_{N\backslash\{k\}}) = (1, p')$.
Now, by using a similar argument as the one in Step 2 by replacing agent 1 by k, we can show that $f_k(\underline{s}) = (1, p')$ which is a contradition to Step 2.
Step 3.2. Agent $k \in N\backslash\{1\}$ gets the good at s and $s_k = \underline{s}_k$.
We obtain a contradiction using an argument similar to the one in Step 3.1.
Thus, agent 1 gets the good at any s and pays p, which ends the proof. ■

Example 8 (continued)
The following Lemma 2 is used in the proof of Proposition 11.

Lemma 2 *For all $s \in \Theta$, $R_k(s'_k, s_{-k})$ is a y-monotonic transform of $R_k(s)$ for all $s'_k > s_k$, $k \in N$ and $y \in A$ such that $y_k = (1, p)$, $p \geq 0$.*

Proof. Take $s \in \Theta$, $k \in N$ and $y \in A$ such that $y_k = (1, p)$, $p \geq 0$ and $s'_k > s_k$. Since g_k is non-decreasing in s_k, $g_k(s'_k, s_{N\backslash\{k\}}) \geq g_k(s)$ which means that agent k values the good in signal profile $(s'_k, s_{N\backslash\{k\}})$ at least as under profile s. Thus, $(1, p)$ weakly improves its position in $R_k(s'_k, s_{N\backslash\{k\}})$ compared to its position in $R_k(s)$. Formally, $R_k(s'_k, s_{N\backslash\{k\}})$ is a y-monotonic transform of $R_k(s)$. ■

Proposition 11 *If Θ is such that for each $i \in N$, g_i is as described in Example 8, then Θ is partially knit.*

Proof. Take any two pairs (x,θ), $(z,\widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta,z,x) \neq \emptyset$, $\#C(\theta,z,x) = 2$. Some agent must get the good either in x or in z, otherwise $\overline{C}(\theta,z,x) = \emptyset$.

First, assume that the same agent i gets the good both in x and in z. Define $\theta' = (\max\{s_i,\widetilde{s}_i\},\ \min\{s_j,\widetilde{s}_j\})$, $S = \widetilde{S} = \{\max\{s_i,\widetilde{s}_i\},\ \min\{s_j,\widetilde{s}_j\}\}$ where $I(S) = I(\widetilde{S}) = \{i,j\}$. Note that for step $h = 1$, either $s_{i(S,1)} = s_{i(\widetilde{S},1)} = s_i$ if $s_i > \widetilde{s}_i$ or $s_{i(S,1)} = s_{i(\widetilde{S},1)} = \widetilde{s}_i$ if $s_i < \widetilde{s}_i$. Thus, either because there is no signal change or by Lemma 2, we obtain that $R_i(m^1(\theta,S))$ is an x-monotonic transform of $R_i(m^0(\theta,S))$ and $R_i(m^1(\widetilde{\theta},\widetilde{S}))$ is an z-monotonic transform of $R_i(m^0(\widetilde{\theta},\widetilde{S}))$. Note that for step 2, either $s_{i(S,h)} = s_{i(\widetilde{S},h)} = s_j$ if $s_j < \widetilde{s}_j$ or $s_{i(S,h)} = s_{i(\widetilde{S},h)} = \widetilde{s}_j$ if $s_j > \widetilde{s}_j$. Thus, either because there is no signal change or by Lemma 1, we obtain in step 2 that $R_j(m^2(\theta,S))$ is an x-monotonic transform of $R_j(m^1(\theta,S))$ and $R_j(m^2(\widetilde{\theta},\widetilde{S}))$ is a z-monotonic transform of $R_j(m^1(\widetilde{\theta},\widetilde{S}))$. Thus, the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Second, suppose that different agents get the good in x and z. Without loss of generality, say that agent 1 gets the good in x while agent 2 gets it in z. Thus, alternatives x and z are such that $x_1 = (1,p_x)$, $z_1 = (0,0)$, $x_2 = (0,0)$, $z_2 = (1,p_z)$.
Now, we consider three cases, and for each one we define θ' and the sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.
Case 1. $\theta = (0,1)$.
The conditions $\overline{C}(\theta,z,x) \neq \emptyset$ and $C(\theta,z,x) = N$ are satisfied since $p_x > l$ and $p_z > l$. For any $\widetilde{\theta}$ define $\theta' = \widetilde{\theta}$. If $\widetilde{\theta} = (1,1)$, let $S = \{\theta_{i(S,1)} = 1\}$, $I(S) = \{1\}$, if $\widetilde{\theta} = (0,0)$, let $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{2\}$, and if $\widetilde{\theta} = (1,0)$, let $S = \{\theta_{i(S,1)} = 1, \theta_{i(S,2)} = 0\}$, $I(S) = \{1,2\}$. By applying Lemma 2, Lemma 1 or both, respectively, we prove that the passage from θ to $\widetilde{\theta} = \theta'$ through S is x-satisfactory.
Case 2. $\theta = (1,1)$.
For conditions $\overline{C}(\theta,z,x) \neq \emptyset$ and $C(\theta,z,x) = N$ to hold we must have either $p_x > m$ and $p_z \leq m$, or $p_z < m$ and $p_x \geq m$. Suppose that the former holds. Otherwise, a similar proof would follow.
If $\widetilde{\theta} = (0,1)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, and observe that $R_1(m^1(\theta,S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$ and $p_z \leq m$.
If $\widetilde{\theta} = (1,0)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{2\}$, and observe that $R_2(m^1(\theta,S))$ is an x-monotonic transform of $R_2(\theta)$ by Lemma 1.
If $\widetilde{\theta} = (0,0)$, let $\theta' = (0,1)$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 1\}$, $I(\widetilde{S}) = \{2\}$. Again, observe that $R_1(m^1(\theta,S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$ and $p_z \leq m$. Moreover, $R_2(m^1(\widetilde{\theta},\widetilde{S}))$ is a z-monotonic transform of $R_2(\widetilde{\theta})$ since $l < p_z \leq m$.
Case 3. $\theta = (0,0)$ and $\theta = (1,0)$.
For both θ, $g_2(\theta) = l$. Since $2 \in C(\theta,z,x)$ then $p_z \leq l$, being a contradiction to our hypothesis in Example 8. Thus, this case must not be considered to check partial knitness.

Third, the last remaining possibility is one where in only one of the two alternatives, x or z, some agent gets the good. Without loss of generality, suppose that agent 1 gets the good in x. Note that for conditions $\overline{C}(\theta,z,x) \neq \emptyset$ and $C(\theta,z,x) = N$ to hold, for any $\theta \in \Theta$, $1 \in \overline{C}(\theta,z,x)$ since $2 \in C(\theta,z,x)$.

Now, we consider four cases, and for each one we define θ' and the sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Case 1. $\theta = (0,1)$.

Since $1 \in \overline{C}(\theta, z, x)$, $p_x > l$ must be satisfied. For any $\widetilde{\theta}$ define $\theta' = \widetilde{\theta}$. If $\widetilde{\theta} = (1,1)$, let $S = \{\theta_{i(S,1)} = 1\}$ and $I(S) = \{1\}$, if $\widetilde{\theta} = (0,0)$, let $S = \{\theta_{i(S,1)} = 0\}$ and $I(S) = \{2\}$, and if $\widetilde{\theta} = (1,0)$, let $S = \{\theta_{i(S,1)} = 1, \theta_{i(S,2)} = 0\}$ and $I(S) = \{1,2\}$. By applying either Lemma 2, Lemma 1 or both consecutively in this order, we prove that the passage from θ to $\widetilde{\theta} = \theta'$ through S is x-satisfactory.

Case 2. $\theta = (1,1)$.

Since $1 \in \overline{C}(\theta, z, x)$, $p_x > m$ must be satisfied.

If $\widetilde{\theta} = (0,1)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, and observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$.

If $\widetilde{\theta} = (1,0)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 1\}$, $I(S) = \{2\}$, and observe that $R_2(m^1(\theta, S))$ is an x-monotonic transform of $R_2(\theta)$ by Lemma 1.

If $\widetilde{\theta} = (0,0)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0, \theta_{i(\widetilde{S},2)} = 0\}$, $I(S) = \{1,2\}$. Again, observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$. Moreover, $R_2(m^2(\theta, S))$ is an x-monotonic transform of $R_2(m^1(\theta, S))$ by Lemma 1.

Case 3. $\theta = (0,0)$.

Since $1 \in \overline{C}(\theta, z, x)$, $p_x > l$ must be satisfied.

If $\widetilde{\theta} = (0,1)$, let $\theta' = \theta$ and define $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0\}$, $I(\widetilde{S}) = \{2\}$, and observe that $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(\widetilde{\theta})$ by Lemma 1.

If $\widetilde{\theta} = (1,0)$, let $\theta' = \theta$ and define $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0\}$, $I(\widetilde{S}) = \{1\}$, and observe that $R_1(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_1(\widetilde{\theta})$ by Lemma 1.

If $\widetilde{\theta} = (1,1)$, let $\theta' = \theta$ and define $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0, \theta_{i(\widetilde{S},2)} = 0\}$, $I(S) = \{2,1\}$, and observe that, by Lemma 1, $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(\widetilde{\theta})$ and $R_1(m^2(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_1(m^1(\widetilde{\theta}, \widetilde{S}))$.

Case 4. $\theta = (1,0)$.

Since $1 \in \overline{C}(\theta, z, x)$, $p_x > h$ must be satisfied.

If $\widetilde{\theta} = (0,0)$, let $\theta' = \widetilde{\theta} = (0,0)$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, and observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > h$.

If $\widetilde{\theta} = (0,1)$, let $\theta' = (0,0)$ and define $S = \{\theta_{i(S,1)} = 0\}$ and $I(S) = \{1\}$, $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0\}$ and $I(\widetilde{S}) = \{2\}$. Observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > h$. Moreover, $R_2(m^2(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ by Lemma 1.

If $\widetilde{\theta} = (1,1)$, let $\theta' = (0,0)$ and define $S = \{\theta_{i(S,1)} = 0\}$ and $I(S) = \{1\}$, $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0, \theta_{i(\widetilde{S},2)} = 0\}$ and $I(\widetilde{S}) = \{1,2\}$. Again, observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$. Moreover, $R_1(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_1(\widetilde{\theta})$ and $R_2(m^2(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ by Lemma 1.

The proof of Proposition 11 ends. ■

Lemma 3 *If Θ is such that for each $i \in N$, g_i is as described in Example 8, $f_{p,p'}$ is non-constant, ex post incentive compatible, and respectful.*

Proof. By definition $f_{p,p'}$ is not constant. To show that $f_{p,p'}$ is ex post incentive compatible we first observe that agent 1 can never strictly gain by deviating from any $s \in \Theta$. For any $s_2 \in \Theta_2$, since $g_1(0, s_2)) = l$, $g_1(1, s_2) \in \{m, h\}$, and $p \in (l, m)$, then $f_1(0, s_2) P_1(0, s_2) f_1(1, s_2)$ and $f_1(1, s_2) P_1(1, s_2) f_1(0, s_2)$ holds where $f_1(0, s_2) = (0, 0)$ and $f_1(1, s_2) = (1, p)$. Similarly, we can show that agent 2 can never strictly gain by deviating from any $s \in \Theta$. For any $s_1 \in \Theta_1$, since $g_2(s_1, 0) = l$, $g_2(s_1, 1) \in \{m, h\}$, and $p' \in (l, m)$, then $f_2(s_1, 0) R_2(s_1, 0) f_2(s_1, 1)$ and $f_2(s_1, 1) R_2(s_1, 1) f_2(s_1, 0)$ where $f_2(s_1, 0) = (0, 0)$ and $f_2(s_1, 1) \in \{(0, 0), (1, p')\}$. To check respectfulness, observe that agent 1 is not indifferent between any pair of outcomes obtained when she is the only one changing types. Concerning agent 2, observe that the same holds if $s_1 = 0$. For $s_1 = 1$, $f(1, 0) = f(1, 1)$. Thus, respectfulness holds. ■

Related to Remark 2

Example 9. For simplicity, let $N = \{1, 2\}$, $S_i = \{0, 1\}$ for all $i \in N$ and $l, m, h \in \mathbb{R}_+$ with $0 = l < m < h$. The agent's preference formation rule is defined as in the general framework but will now be based on a different auxiliary function that takes three possible values, low, medium and high.

More formally,

$$g_i(s) = \begin{cases} l & \text{if} \quad s_i = 0 \\ m & \text{if} \quad s_i = 1 \text{ and } s_j = 0, \\ h & \text{if} \quad s_i = 1 \text{ and } s_j = 1. \end{cases}$$

Observe that for each agent i, g_i satisfies (a) and the following condition:
(e) g_i is non-decreasing in s_j, for all $j \in N \backslash \{i\}$.

Condition (e) establishes that the valuation of the good by agent i depends positively on other agents' signals.

Now, we assert that the domain of types in Example, 9 is neither knit nor partially knit. To do so, we define below a non-constant mechanism on this domain that is ex post incentive compatible and respectful but it is not ex post group incentive compatible. Therefore, by Theorem 2, the domain is not partially knit.

A mechanism $f_{h,m}$ is such that agent 1 gets the good and pays h if the signal of agent 2 is 0 or both agents' signals are 1 and agent 2 gets the good and pays m otherwise. Formally, for $\theta \in \{0, 1\}^2$,

$$f_{h,m}(\theta) = \left\{ \begin{array}{c} ((1, h), (0, 0)) \text{ if } s_2 = 0, \\ ((0, 0), (1, m)) \text{ if } s_1 = 0,\ s_2 = 1, \text{ and} \\ ((1, h), (0, 0)) \text{ if } s_1 = 1 = s_2 \end{array} \right\}.$$

Lemma 4 *If Θ is such that for each $i \in N$, g_i is as described in Example 9, $f_{h,m}$ is non-constant, ex post incentive compatible, and respectful but violates ex post group incentive compatibility.*

Proof. To check ex post incentive compatibility just observe that no single agent can strictly gain by unilaterally deviations. To check respectfulness, we need to consider s and s' such that $f(s) \neq f(s')$, where only one agent changes her type and remains indifferent. Two cases need to be checked. First, let $s = (1, 1)$, $s' = (0, 1)$. Observe that neither $R_1(0, 1)$ is a $f(1, 1)$-monotonic transform of $R_1(1, 1)$ nor $R_1(1, 1)$ is a $f(0, 1)$-monotonic transform of $R_1(0, 1)$. Second, let $s = (0, 1)$ and $s' = (0, 0)$. Again, neither $R_2(0, 0)$ is a $f(0, 1)$-monotonic transform of $R_2(0, 1)$ nor $R_2(0, 1)$ is a $f(0, 0)$-monotonic transform of $R_2(0, 0)$. Thus these cases do not need to be considered and respectfulness holds. To check that $f_{h,m}$ violates ex post group incentive compatility, consider $s = (1, 1)$, $C = N$, $s'_C = (0, 1)$. Note that $(0, 0) = f_1(0, 1)I_1(1, 1)f_1(1, 1) = (1, h)$ and $(1, m) = f_2(0, 1)P_2(1, 1)f_2(1, 1) = (0, 0)$ which means that coalition N can ex post profitably deviate under mechanism f at $s \in \Theta$ via s'_C. ■